Common Qualifying Exam Part B
Computational and Applied Math
January 2015

Solve three out of four problems

(1). Find the eigenvalues and the corresponding eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0$$

subject to the following boundary conditions

(a). $y(0) = y(L) = 0$.
(b). $y'(0) = y'(L) = 0$.
(c). $y(0) = y'(L) = 0$.
(d). $y(0) = y(L) + y'(L) = 0$. 
(2). Find the general solution of the following differential equation

\[ y^{(4)} - 2y^{(3)} + y'' = 1 + xe^x + \sin x \]
(3). Let $A \in \mathbb{R}^{m \times n}$ be a matrix with rank $n - 1$, where $m \geq n$. Suppose $AP = QR$ is the QR factorization of $A$ with column pivoting, where $P$ is a permutation matrix, such that the diagonal entries of $R$ are non-increasing in magnitude, i.e., $|r_{11}| \geq |r_{22}| \geq \cdots \geq |r_{nn}|$. Ignore the effect of rounding errors.

(a). Show that $r_{nn} = 0$.

(b). Show that $r_{n-1,n-1} \neq 0$. 
(4). Suppose $A \in \mathbb{R}^{n \times n}$ is normal.

(a). Show that the singular values of $A$ are equal to the magnitudes of the eigenvalues of $A$.

(b). Let $x$ be an eigenvector corresponding to a complex eigenvalue $\lambda_i = \alpha_i + i\beta_i$, where $\beta_i \neq 0$. Show that the real and complex parts of $x$, i.e., $\text{Re}(x)$ and $\text{Im}(x)$, are orthogonal to each other.