Qualifying Exam (January 2014): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.
Do 2 out of problems 1,2,3.
Do 2 out of problems 4,5,6.
Do 3 out of problems 7,8,9,10,11,12,13,14.

All problems are weighted equally. **On this cover page write which seven problems you want graded.**

problems to be graded:

_________________________________________________________________________________________

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Name (PRINT CLEARLY), ID number**

_________________________________________________________________________________________

Signature
1). Consider the standard form polyhedron $P = \{ x \mid Ax = b, \ x \geq 0 \}$, and an objective function $\max cx$. (As usual, assume that $A$ is an $m$ by $n$ matrix whose rows are linearly independent, $b$ is a vector of dim $m$, and $c$ and $x$ are vectors of dim $n$.) For each of the following statements, state whether it is true or false. If true, give a short proof; otherwise, provide a counterexample or explanation.

(a). The set of all optimal solutions is bounded.
(b). If there is more than one optimal solution, then there are at least two Basic Feasible solutions that are optimal.
(c). If the LP is degenerate, it may have an infinite number of BFS’s.
(d). If there is more than one optimal solution, then there are infinitely many optimal solutions.
(e). Let $x'$ be a feasible solution with exactly $m$ variables that are positive, then $x'$ is a BFS.
(f). At every optimal solution, no more than $m$ variables can be positive.

2). Prove: For all matrices $A$ and vectors $b$, exactly one of the following two alternatives holds:

(I) There exists $x \geq 0$ such that $Ax = b$,

or

(II) There exists $y$ such that $yA \leq 0$ and $yb = 1$.

3). The following LP was solved (using the big M method) and the optimal tableau is given below. $e_1$ and $e_2$ are the excess variables subtracted from the first and second constraints, and $a_i$ is the artificial variable of the $i$th constraint.

$$\text{max} \quad z = 4x_1 + x_2$$
$$\text{s.t.} \quad 3x_1 + x_2 \geq 6$$
$$\quad 2x_1 + x_2 \geq 4$$
$$\quad x_1 + x_2 = 3$$
$$\quad x_1, x_2 \geq 0$$

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(a). Find the dual of this LP and its optimal solution (the objective value and the value of the dual variables). Use the tableau – do not solve from scratch!
(b). Find the range of values of the objective function coefficient for $x_2$ for which the current basis remains optimal.
(c). Find the range of values of $b_1$ (the right hand side of the first constraint) for which the current basis remains optimal.
(d). We wish to add to the LP the constraint $x_2 \geq 1.5$, for which the current optimal solution is not feasible. Set up a tableau and use the dual Simplex method to find the new optimal solution.
(e). A new variable is introduced, giving the following LP: Find the new optimal solution using sensitivity analysis (do not solve from scratch!).

2
max $z = 4x_1 + x_2 + 20x_3$

s.t. $\begin{align*}
3x_1 + x_2 & \geq 6 \\
2x_1 + x_2 + x_3 & \geq 4 \\
x_1 + x_2 + x_3 & = 3 \\
x_1, x_2 & \geq 0
\end{align*}$

4). Let $Z_1, Z_2, \ldots$ be a sequence of i.i.d. random variables with mean 0 and finite variance $\sigma^2$. Define $T_n = \sum_{i=1}^{n} Z_i$. Let $M$ be a finite stopping time with respect to the $Z_i$ sequence such that $E[M] < \infty$. Show that $Var(T_M) = E[M] \sigma^2$.

5). A population begins with a single individual. In each generation, each individual in the population dies with probability 0.5 or doubles with probability 0.5. Let $X_n$ denote the number of individuals in the $n$th generation. Find the variance of $X_n$.

6). Customers arrive at a bank according to a Poisson process of rate $\lambda$ customers/hour. Let $X(t)$ be the number of customers that have arrived up to time $t$. Let $T_n$ be the arrival time of the $n$-th arrival, $n = 1, 2, \ldots$. Find the conditional mean $E[T_6 | X(t) = 3]$.

7). For each of the computations below indicate how efficiently one can perform the calculation, in terms of $O(\ldots)$ notation (e.g., $O(n)$, $O(\log n)$, $O(n^3)$, $O(n \log n)$). Try to give the best upper bound possible in terms of $n$ and $h$ (the number of convex hull vertices, if relevant). **FOR FULL CREDIT, YOU MUST GIVE SOME BRIEF JUSTIFICATION.**

(a). You are given 2 disjoint simple polygons, $A$ (having $n^2$ vertices) and $B$ (having $n^3$ vertices), each specified by its vertex set in ccw order around the polygon, with polygon $A$ contained in the first quadrant, and $B$ within the second quadrant. You are to (i) triangulate each polygon, and (ii) compute the convex hull, $CH(A \cup B)$, of their union.

(b). You are given 3 disjoint simple polygons, $A$ (having $n$ vertices), $B$ (having $n^2$ vertices), and $C$ (having $n^3$ vertices), each specified by its vertex set in ccw order around the polygon. You are to (i) triangulate each polygon, and (ii) compute the convex hull, $CH(A \cup B \cup C)$, of their union.

(c). You are given 3 simple polygons, $A$ (having $n$ vertices), $B$ (having $n^2$ vertices), and $C$ (having $n^3$ vertices), each specified by its vertex set in ccw order around the polygon. You are not sure if they are disjoint or not, so you need to find out if they are pairwise disjoint.

8). Obtain the point guard number, $g(P)$, for the polygon $P$ below, allowing guards to be placed at any point (interior or boundary) of the polygon. **Justify your answer:**

(i). Clearly mark your choice of point guards;

(ii). Give a convincing argument that fewer point guards cannot suffice.
(b). Now obtain the vertex guard number, \( g_V(P) \), for \( P \). **Justify your answer:**

(i). Clearly mark your choice of vertex guards;

(ii). Give a convincing argument that fewer vertex guards cannot suffice.

(c). Give an example of a simple \( n \)-gon (with \( n > 8 \)) for which \( g_V(P) \) is significantly larger than \( g(P) \). Explain.
9). Consider a Markov Decision Process (MDP) with the state space \( X = \{1, 2, 3\} \) and action sets \( A(1) = A(2) = A(3) = \{a^1, a^2\} \). Let the transition probabilities be \( p(x|y, a^1) = \frac{1}{3} \), where \( x, y \in X \), and \( p(x|y, a^2) = \frac{1}{2} \), where \( x, y \in X \) and \( x \neq y \). The one step rewards are \( r(1, a^1) = r(3, a^2) = 1 \), \( r(1, a^2) = r(2, a^1) = r(2, a^2) = 2 \), and \( r(3, a^1) = 3 \). The goal is to maximize expected average rewards per unit time over the infinite horizon. Write the primal and dual linear programs (LPs) for this MDP.

10). Is it true that the value iteration algorithm always converges to an optimal policy for a negative dynamic programming program? (A negative dynamic programming problem is a total-reward MDP with nonpositive reward functions.) Provide a proof or a counterexample.

11). We are given a directed graph \( G = (N, A) \), a source node \( s \) and a sink node \( t \) with (nonnegative) capacities on the arcs \( u_{ij} \). We consider the max flow problem on this graph. A most vital arc is defined as an arc whose deletion causes the largest decrease in the maximum flow value \( v \). A least vital arc is defined as an arc whose deletion causes the smallest decrease in the maximum flow value \( v \).

Prove or give a counterexample:
(a) A most vital arc is an arc with the maximum value of \( x_{ij} \).
(b) A most vital arc is an arc with the maximum value of \( x_{ij} \) among arcs belonging to some minimum cut.
(c) An arc that does not belong to some minimum cut cannot be a most vital arc.
(d) Any arc with the \( x_{ij} = 0 \) in any maximum flow is a least vital arc.
(e) A least vital arc is an arc with minimum value \( x_{ij} \) in a maximum flow.

12). (a) Let \( M_1 \) and \( M_2 \) be two arbitrary matchings in a bipartite graph \( G = (N_1 \cup N_2, E) \). Let the nodes of \( N_1 \) that are matched in \( M_1 \) be \( X \ (X \subseteq N_1) \), and the nodes of \( N_2 \) that are matched in \( M_2 \) be \( Y \ (Y \subseteq N_2) \). Show that there exists some matching \( M \) in which \( X \cup Y \) are all matched.

(b) Let \( G = (N, E) \) be a graph, \( N' \subseteq N \) a subset of the nodes that is matched in some matching \( M \). Show that there exists a maximum cardinality matching \( M^* \) in which each node of \( N' \) is matched.

13). Consider a single-server queueing system with i.i.d. service times. Let \( \mu \) be the mean service time, \( W_n \) be the delay of the \( n \)th customer in system, and \( D_n \) be the delay of the \( n \)th customer in queue. Clearly, \( D_n = W_n - S_n \), where \( S_n \) is the service time of the \( n \)th customer. Therefore, \( E[D_n] = E[W_n] - \mu \). If we want to estimate \( E[D_n] \) using simulation, should we (a) directly use the simulated data to construct an estimator for \( E[D_n] \); or (b) use the simulated data to determine \( W_n \), and then use this quantity minus \( \mu \) as an estimate of \( E[D_n] \)? State which option you prefer and explain why. Repeat the question if we want to estimate \( E[W_n] \).

14). Give an algorithm for generating random variates from the following cumulative distribution function

\[
F(x) = \begin{cases} 
\frac{1-e^{-2x}+2x}{3}, & \text{if } 0 < x \leq 1 \\
\frac{3-e^{-2x}}{3}, & \text{if } 1 < x < \infty.
\end{cases}
\]