Quantitative Finance Qualifying Exam
2014 Winter

INSTRUCTIONS
You have 4 hours to do this exam.

Reminder: This exam is closed notes and closed books. No electronic devices are permitted.
Phones must be turned completely off for the duration of the exam.

PART 1: Do 2 out of problems 1, 2, 3.
PART 2: Do 2 out of problems 4, 5, 6.
PART 3: Do 2 out of problems 7, 8, 9.
PART 4: Do 2 out of problems 10, 11, 12.

All problems are weighted equally.
On this cover page write which eight problems you want graded.

Problems to be graded:

__________________________________________
Academic integrity is expected of all students at all times,
whether in the presence or absence of members of the faculty.
Understanding this, I declare that I shall not give, use,
or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

__________________________________________
Signature

Stony Brook University
Applied Mathematics and Statistics
Problem 1
Suppose the market has a risk-free asset with return $r_f$, and $n$ risky assets. When short selling is allowed, the minimum-variance portfolio with the expected return $\mu_*$ can be computed by solving the optimization problem:

$$
\min_{\omega} \omega^T \Sigma \omega \quad \text{subjected to} \quad \omega^T \mu + (1 - \omega^T) r_f = \mu_*,
$$

where $\mu$ and $\Sigma$ are the mean and covariance matrix of returns of $n$ assets, and $1 = (1, 1, ..., 1)^T$.

Show that the solution for $\omega$, when $\Sigma$ is nonsingular and $\mu \neq r_f 1$, is

$$
\omega_{\text{eff}} = \frac{\mu_* - r_f}{(\mu - r_f 1)^T \Sigma^{-1} (\mu - r_f 1)} \Sigma^{-1} (\mu - r_f 1).
$$
Problem 2
Suppose in a complete market, the stock price \((S_t)_{t \geq 0}\) is modeled by
\[
dS_t = \mu S_t dt + \sigma S_t dB_t,
\]
where \((B_t)_{t \geq 0}\) is a standard Brownian motion and \(\mu\) and \(\sigma\) are positive parameters. The risk free rate of return is \(r > 0\). An innovative financial institution has announced that it will trade a security with payoff at time \(T\) given by
\[
g(S_T) = \log(S_T).
\]
(a) Deduce the Black-Scholes equation under the no-arbitrage assumption.
(b) Use risk-neutral valuation to find the security price at time \(t\), \(0 \leq t \leq T\).
(c) Confirm that the price satisfies the Black-Scholes equation.
Problem 3
Heat plays Spurs in NBA final, a series of 7 games, whoever wins 4 games first wins the championship. You want to bet 100 dollars that Heat will win the series, in which case you will receive 200 dollars, or nothing if they lose. However the broker only allows bets on individual games. You can bet $X$ dollars on any individual game (before the game begins), and receive $2X$ dollars if Heat wins and 0 dollars if they lose.

In order to achieve the desired pay-out, how you are going to bet on the first game?
Problem 4
Consider a portfolio consisting of two stocks, and let $R_1$, and $R_2$ be continuous random variables of the returns of the two stocks. Let $F(x, y)$ be the joint distribution of $(R_1, R_2)$. Define copula and calculate probability $\mathbb{P}[R_1 \leq -VaR_\alpha(R_1) \text{ and } R_2 \leq -VaR_\alpha(R_2)]$ using copula, where $VaR_\alpha(R_n)$ is the value at risk of $R_n$ at $\alpha$ significance level for $n = 1, 2$. 
Problem 5

Consider the following ARMA(1,1)-GARCH(1,1) model for the log-return of a portfolio,

\[
    r_t = c + ar_{t-1} + b\sigma_{t-1}\epsilon_{t-1} + \sigma_t \epsilon_t,
\]

\[
    \sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \sigma_{t-1}^2 + \beta_1 \sigma_{t-1}^2,
\]

where \( r_t \) is the log-return, \( \sigma_t \) is the conditional standard deviation, and the innovations \( \epsilon_t \) are independent standard normal random variables.

Based on the historical data set \( \{ r_t, 1 \leq t \leq T \} \), which is the daily log-return of one’s portfolio in the last five years, let us estimate tomorrow’s Value-at-Risk. Assume that the weight of each asset in this portfolio remains the same all the time.

(a) Describe how to estimate \( VaR_{99\%}(r_{T+1}) \), the 99% Value-at-Risk at time \( T + 1 \).

(b) Explain how to derive an empirical 95% confidence interval of the estimator you get in part (a).
Problem 6
Using generalized Merton’s firm value model, dynamic of a firm value is given by the following equation under risk neutral measure:

\[ dV_t = r_t V_t dt + \sigma V_t dW_t^{(1)}, V_0 = v, \]

where \((r_t)_{t \geq 0}\) is the risk free rate of return process, which follows Cox-Ingersol-Ross model (under risk neutral measure):

\[ dr_t = (a - br_t) dt + \sigma \sqrt{r_t} dW_t^{(2)}, r_0 = r. \]

\(a, b, \sigma, \sigma, v,\) and \(r\) are all positive constant. \(W_t^{(1)}\) and \(W_t^{(2)}\) are both standard Brownian Motions with \(dW_t^{(1)} dW_t^{(2)} = \rho dt\), where \(-1 \leq \rho \leq 1\). Let \(\bar{B}(t, T)\) be the price of a defaultable zero coupon bond at time \(t\) with the maturity \(T\) and the principal \(D\). The defaultable bond payoff at time \(T\) is given by

\[ \bar{B}(T, T) = \begin{cases} D & V_T > D, \\ V_T & V_T \leq D. \end{cases} \]

Describe Monte Carlo analysis to determine the risk neutral price \(\bar{B}(0, T)\) at time \(t = 0\) using \(N\) trials.
Problem 7
For a European call expiring at time $T$ with strike price $K$, paying no dividends, the Black-Scholes price at time $t$, if the time-$t$ stock price is $x$, is

$$c(t, x) = xN(d_+(T-t, x)) - Ke^{-r(T-t)}N(d_-(T-t, x)).$$

Here

$$d_+(\tau, x) = \frac{1}{\sigma\sqrt{\tau}} \left[ \log \frac{x}{K} + \left( r + \frac{1}{2}\sigma^2 \right) \tau \right],$$

and $N(y)$ is the cumulative standard normal distribution. Prove that option delta $\Delta = \frac{\partial c}{\partial x}$ and the option theta $\Theta = \frac{\partial c}{\partial t}$ are given by

$$\begin{align*}
\Delta(t, x) &= N(d_+(T-t, x)) \\
\Theta(t, x) &= -rKe^{-r(T-t)}N(d_-(T-t, x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t, x)).
\end{align*}$$
Problem 8
Let $W$ be Brownian motion on the interval $[0, T]$, $T > 0$, and define the Brownian bridge $X$ by

$$X(t) = W(t) - \frac{t}{T}W(T), \quad t \in [0, T].$$

Let $s, t$ be such that $0 \leq s, t \leq T$. Compute the covariance of $X(s)$ and $X(t)$. 
Problem 9

Derive the solution $R$ to the Vasicek interest rate model:

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t), \quad R(0) = R_0,$$

where $\alpha, \beta, \sigma, R_0 > 0$ are constants, and $W$ is Brownian motion.
**Problem 10**

Consider a European call option with parameters as follows: current stock price $S_0$, strike $K$, risk-free rate $r$, volatility rate $\sigma$, and time to maturity $T$ years. Assuming a geometric Brownian motion for the stock price process $S_t$, use the delta-normal valuation to compute the 95% VaR over a horizon of 3 days for a long position of a European call.
Problem 11

Consider the following ARMA(1,1)-GARCH(1,1) model for daily return $r_t$,

$$r_t - \phi r_{t-1} = u_t + \psi u_{t-1}, \quad u_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2,$$

where $\phi, \psi \in (0, 1)$, $\omega > 0, \alpha, \beta \in (0, 1)$ and $\alpha + \beta < 1$. What is the 99% 2-day VaR of a long position at time $t$?
Problem 12

Consider the Clayton copula with $\theta = -1/2$, the copula can be written as $C(u_1, u_2) = \max\{(\sqrt{u_1} + \sqrt{u_2} - 1)^2, 0\}$. Show that its Kendall’s tau correlation is $-1/3$. 