AMS Common Exam - Part A, June 2014

Name: ________________________________ & ID Num. __________________
Part A: _____ / 75
Part B: _____ / 75 & Total: _____ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose THREE questions to answer from EACH section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!
Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Given the linear transformation $F : \mathbb{R}^4 \to \mathbb{R}^3$ defined by:

   $$F(x, y, z, t) = (x - y + s + t, x + 2z - t, x + y + 3z - 3t)$$

   (a) . Find the orthonormal basis of $\text{Im}(F)$.

   (b) . Find the orthonormal basis of $\text{Ker}(F)$
2. Given the linear system

\[
\begin{align*}
2x_1 - \lambda x_2 - x_3 &= 1 \\
\lambda x_1 - x_2 + x_3 &= 2 \\
4x_1 + 5x_2 - 5x_3 &= -1
\end{align*}
\]

(a) For what value of \( \lambda \), the system has no solution.

(b) For what value of \( \lambda \), the system has a unique solution.

(c) For what value of \( \lambda \), the system has infinite solutions; find the formula of the general solution.
3. Find the determinant of the matrix $A$.

$$A = \begin{pmatrix}
1 + x_1^2 & x_1x_2 & \cdots & x_1x_n \\
x_2x_1 & 1 + x_2^2 & \cdots & x_2x_n \\
\vdots & \vdots & \ddots & \vdots \\
x_nx_1 & x_nx_2 & \cdots & 1 + x_n^2
\end{pmatrix}$$
4. Let $V$ be a finite-dimensional linear vector space and $U$ be a linear vector space (not necessarily of finite dimension). Let $T : V \rightarrow U$ be a linear mapping, prove that

$$dim(V) = dim(Ker(T)) + dim(Im(T))$$

where $Ker(\cdot)$ and $Im(\cdot)$ represent kernel and image.
Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Find the value of following formulas:

(a) \[ \int_{0}^{\infty} \frac{dx}{x^4 + 4} \]

(b) \[ \int_{0}^{\pi/2} \frac{\sin^m x}{\sin^m x + \cos^m x} dx, \ m \in R \]
2. Prove the following conclusions

(a). When $|x| \geq 1$

$$2 \arctan x + \arcsin \frac{2x}{1 + x^2} = \pi \text{sign} (x)$$

(b). $|\arctan a - \arctan b| \leq |a - b|$
3. Calculate the volume of the solid enclosed by

\[
\left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)^2 = \frac{x}{h}.
\]
4. Find the extreme value of following function

\[ u(x_1, \cdots, x_n) = x_1x_2^2 \cdots x_n^n(1 - x_1 - 2x_2 - \cdots - nx_n) \]

where \( x_1, \cdots, x_n > 0. \)