AMS Common Exam - Part B (CAM), June 2014

This component of the exam (Part B) consists of four problems, choose **THREE** of the four questions to answer. If you do four, the three problems with the lower scores will be counted!

(1). Given the following matrix

\[
A = \begin{pmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{pmatrix}.
\]

(a). Perform QR decomposition on \( A \) using the Gram-Schmit algorithm.

(b). Perform Householder triangularization on \( A \).

(c). Perform the LU decomposition (Gauss elimination) on \( A \).

(d). Perform the Cholesky factorization of the matrix \( A \).

If you can make three right, you will have full credit (doing four will not add number of points, but may increase your safety margin).
(2). (a). Show that if \( x \) is an \( n \)-vector, then

\[ ||x||_\infty \leq ||x||_2 \leq \sqrt{n} ||x||_\infty. \]

When will the equality apply? Can the equality be applied to the above two inequalities simultaneously?

(b). For a given matrix

\[
A = \begin{pmatrix}
10 & 1 & 0 \\
1 & 10 & 1 \\
0 & 1 & 10
\end{pmatrix},
\]

calculate \( ||A||_1, ||A||_\infty \) and estimate \( ||A||_2 \).
(3). Suppose $y_1(x)$, $y_2(x)$, and $y_3(x)$ are homogeneous solutions to a 3rd order, homogeneous linear differential equation. Find Green’s function for the corresponding inhomogeneous equation, and generalize it to n-th order differential equation.
(4). The boundary value problem

\[ \varepsilon y'' + x^2 y' - y = 0, \quad y(0) = y(1) = 1, \]

has a boundary layer with the thickness of the order $\varepsilon^{1/2}$ at $x = 0$. Introduce the boundary layer coordinates, find inner and outer differential equations and their solutions in the lowest order, and perform their match. Write down the uniform solution.