INSTRUCTIONS
You have 4 hours to do this exam.

Reminder: This exam is closed notes and closed books. No electronic devices are permitted. Computers, cell phones, tablets must all be turned off and put away for the duration of the exam. All problems are weighted equally.

PART 1: Do 2 out of problems 1, 2, 3.
PART 2: Do 2 out of problems 4, 5, 6.
PART 3: Do 2 out of problems 7, 8, 9.
PART 4: Do 2 out of problems 10, 11, 12.

For each problem an extra page is provided which you may use for scratch work or to continue your solution. On this cover page write which eight problems you want graded.

Problems to be graded:

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Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number:

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Signature:
Problem 1 (Pricing a Custom European Option)
Consider a cash chooser European option on an underlying security with price $S(t)$ and strike $K$. The holder of the option has the right at expiry to either exercise the option or receive a fixed cash payment of $B$.

- Sketch out the pay-off function of a cash chooser European put at expiry. Clearly show the values $\{S(T), F(T)\}$ of the pay-off function at key inflection points.

- Write out a pricing formula for the price of the chooser option at time $t < T$, $F(S(t), t)$, in terms of vanilla European puts and calls on $S$ and, if necessary, any cash position saved or borrowed. Assume continuous compounding with a risk-free rate of return $r_f$. Use $P(S(t), K)$ and $C(S(t), K)$ to denote the price of a vanilla European put and call on $S$, respectively, at time $t$ with strike price $K$. Show all work in the derivation.
Extra Paper 1
Problem 2 (Transformation of a Stochastic Process)

A security’s price is found to be governed by the following geometric Itô process:

\[ dS(t) = \mu S(t) \, dt + \sigma S(t) \, dW(t) \]

Let \( V(S(t), t) = \ln[S(t)] \). Find the Itô process which governs \( V \). Show all work in the derivation.
Extra Paper 2
Problem 3  (CAPM and the Market Portfolio)

Consider a market composed of securities governed by the Capital Asset Pricing Model (CAPM) with beta vector $\beta = \{\ldots, \beta_i, \ldots\}$, and error variance vector, i.e., the variances to the error term of the CAPM, $\sigma^2 = \{\ldots, \sigma_i^2, \ldots\}$. Further assume that all investors can invest or borrow at the risk-free rate $r_f$ and that all investors are mean-variance optimizers.

(a) What is the expression to proportionality for $x = \{\ldots, x_i, \ldots\}$, the relative allocation (i.e., $1^T x = 1$) of securities in the market portfolio.

(b) Let $\mu_M$ represent the expected market return. Express the expected returns of the assets in the market portfolio in terms of the assets’ $\beta$ and $\sigma^2$ vectors.

(c) Let $\Sigma$ represent the covariance matrix of the assets in the market portfolio. Express this matrix in terms of the assets’ $\beta$ and $\sigma^2$ vectors.
Extra Paper 3
Problem 4 (Copula)

Let $F_X[x]$ be the CDF of a continuous multivariate random variable $X$ with marginals $F_{X_i}[x_i], i = \{1, \ldots, n\}$ and let Uniform[0,1] designate the uniform distribution on the unit interval.

(a) Show that the random variable $U = F_{X_i}[X_i] \sim \text{Uniform}[0,1]$.

(b) Let $G_Y[y]$ be the CDF of a continuous univariate random variable $Y$. Show that the random variable $G_Y^{-1}[U]$ where $U \sim \text{Uniform}[0,1]$ realizes a random variable with the same distribution as $Y$.

(c) Derive the copula associated with $F_X[x]$, i.e., a function $C_X[u], u = \{u_1, \ldots, n\}$ where $C$ is a multivariate CDF with Uniform[0,1] marginals, representing the dependence structure of $F_X[x]$ separate from its marginals.

(d) Let $G_{Y_i}[y_i], i = \{1, \ldots, n\}$ designate the CDFs of continuous univariate random variables $Y_i, i = \{1, \ldots, n\}$. We wish to construct a multivariate distribution $G_Y[y]$ with marginals $G_{Y_i}[y_i]$ and the same dependence structure as $F_X[x]$. Write an expression for $G_Y[y]$ which accomplishes this.
Extra Paper 4
Problem 5 (Elliptical Distribution)

Let Uniform[0, 1] designate the uniform distribution on the unit interval. We wish to construct a bivariate elliptical distribution for a random variable \( U \) based on the uniform distribution with location vector \( \mu \) and positive-definite dispersal matrix \( \Lambda \), i.e., with PDF \( U_2[u] = \text{Elliptical}[\mu, \Lambda, g_2] \) for \( u \in \mathbb{R}^2 \).

(a) Derive an appropriate generator function \( g_2 \) for the function \( U_2 \).
(b) Derive the PDF \( U_2[u] = \text{Elliptical}[\mu, \Lambda, g_2] \).
(c) What are the mean and covariance of the random variable \( U \)?
Extra Paper 5
Problem 6 (Market Portfolio)

Consider a market composed of securities whose returns are Gaussian with mean vector $\mu$ and covariance matrix $\Sigma$. Further assume that all investors can invest or borrow at the risk-free rate $r_f$, that all investors are mean-variance optimizers, and that there are no constraints on short positions. Let $x = \{\ldots, x_i, \ldots\}$ designate the relative allocation of securities in the market portfolio; i.e., $1^T x = 1$ where shorts are represented by negative positions.

(a) Derive the mean-variance quadratic program (MVQP) which expresses an optimal portfolio in which unallocated capital (where $1^T x < 1$) is invested at the risk-free rate and a leveraged portfolio (where $1^T x > 1$) is funded by borrowing at the risk-free rate.

(b) Solve the MVQP above and then derive the solution for the market, i.e., tangent, portfolio. Show all work in derivation.
Problem 7
Consider a European call option with strike $K > 0$ and expiry $T > 0$ on an underlying stock $S(t)$ governed by geometric Brownian motion, with drift $\mu$ and volatility $\sigma > 0$. Assume a constant risk-free rate $r > 0$. Given $S(0) = S_0$, derive the probability at time $t = 0$ that the option expires in the money.
Problem 8
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be independent random variables on a probability space $(\Omega, \mathcal{F}, \text{Prob})$, and $f$ and $g$ Borel measurable functions $f, g : \mathbb{R} \to \mathbb{R}$. Prove that the random variables $f(X)$ and $g(Y)$ are independent.
Problem 9
Consider a multidimensional market model with $m$ stocks $S_i(t)$ driven by $d$-dimensional Brownian motion $[W_1(t),...,W_d(t)]$:

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t), \quad i = 1,...,m.$$ 

The rate-of-return means $\alpha_i(t)$, and volatility matrix elements $\sigma_{ij}(t)$ are stochastic processes adapted to the Brownian motion filtration. Let $B_i(t)$ be given by

$$B_i(t) = \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)} dW_j(u), \quad i = 1,...,m$$

where $\sigma_i(t) = \sqrt{\sum_{j=1}^{d} \sigma_{ij}^2(t)} > 0$. Show that $B_i(t)$ is a Brownian motion, and that the volatility of $S_i(t)$ is $\sigma_i(t)$. 

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Extra Paper 9
Problem 10
Consider a portfolio of $N$ loans of the same amount $M$ that mature at time $T$. Suppose that the probability of default of each loan is $p$ and that the probabilities of default are independent. Compute the ES (Expected Shortfall) of the portfolio at alpha confidence level. Assume $N$ is large and use the normal approximation to the binomial distribution.
Problem 11
We want to test a daily VaR$_{0.95}$ using a sample of 252 daily observations. Compute the 90% confidence interval of the number of violations.
(Notes: recall that violations are days when losses exceed VaR, use the normal approximation to binomial, and recall that if $Z$ is a standard normal variable, $P(Z > 1.65) = 0.05.$)
Problem 12
Suppose that the Expected Shortfall at confidence level $\alpha$ $ES_\alpha$ is the following function:

$$ES_\alpha = 2(1 - \alpha)^{-1/2}.$$ 

Compute the Value at Risk VaR$_\alpha$ at confidence level $\alpha = 0.95$. 
Extra Paper 12