Doctoral Qualifying Examination – Fall 2004
Linear Algebra and Advanced Calculus

DIRECTIONS: Answer 6 questions, 3 from Linear Algebra, 3 from Advanced Calculus. Good luck!

Linear Algebra:

1. Let

\[ A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}. \]

(a) Find the dimension and a basis for the null space of \( A \).
(b) Determine whether the linear system \( Ax = b \) has a unique solution, no solution, or multiple solutions. Justify your answer.

2. Let \( V \) be the vector space over \( \mathbb{R} \) of polynomials of degree \( \leq 2 \).
   (a) Show that \( \langle p, q \rangle \equiv p(-1)q(-1) + p(0)q(0) + p(1)q(1) \) defines an inner product in \( V \). [Recall: A nonzero polynomial of degree \( n \) has at most \( n \) distinct real zeros.]
   (b) Find an orthogonal basis for \( V \).

3. Let \( A \) be a real, skew-symmetric \( n \times n \) matrix (i.e., \( A^T = -A \)). Show that all the eigenvalues of \( A \) are purely imaginary (i.e., if \( \lambda \) is an eigenvalue, then \( \lambda = i\mu \) with \( \mu \) real).

4. Let \( A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \).
   (a) Find a nonsingular matrix \( P \) such that \( D = P^{-1}DP \) is diagonal.
   (b) Find \( f(A) \), where \( f(x) = x^4 - 3x^3 - 6x^2 + 7x \).
   (c) Find a “positive square root” of \( A \), that is, a matrix \( B \) such that \( B^2 = A \) and \( B \) has positive eigenvalues.
Advanced Calculus:

1. Let \( f(x) = \sum_{n=1}^{\infty} \sin^2(\sqrt{n}x)/n^2 \).
   (a) Prove that \( f(x) \) is continuous for all \( x \).
   (b) Evaluate \( \lim_{x \to 0} f(x) \).

2. Is the improper integral \( \int_0^\infty \alpha e^{-\alpha x} \, dx \) uniformly convergent for \( \alpha \geq 0 \)? Justify your answer. (Hint: Consider the limits: \( \lim_{\alpha \to 0^+} \int_0^\infty \alpha e^{-\alpha x} \, dx \) and \( \int_0^\infty \lim_{\alpha \to 0^+} (\alpha e^{-\alpha x}) \, dx \).)

3. Investigate the convergence of the following series. In case of convergence, test for absolute or conditional convergence. Please do not forget to justify your answers.

   (a) \( \sum_{n=0}^{\infty} \left( \frac{-3}{2} \right)^n \)
   (b) \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1} \)
   (c) \( \sum_{n=1}^{\infty} \frac{(-3)^n}{n 5^n} \)

4. Let \( u_1 = 3 \) and \( u_{n+1} = \sqrt{u_n + 1}, \, n = 1, 2, \ldots \).
   (a) Prove that the sequence \( \{u_n\} \) converges.
   (b) Evaluate \( \lim_{n \to \infty} u_n \).