Problems from AMS 504 for Qualify Exam

1. Let \( f, g : [0, 1] \to [0, \infty) \) be continuous functions satisfying

\[
\sup_{0 \leq x \leq 1} f(x) = \sup_{0 \leq x \leq 1} g(x).
\]

Prove that there exists \( t \in [0, 1] \) with \( f^2(t) + 3f(t) = g^2(t) + 3g(t) \).

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be twice differentiable, and suppose that for all \( x \in \mathbb{R} \), \( |f(x)| \leq 1 \) and \( |f''(x)| \leq 1 \). Prove that \( |f'(x)| \leq 2 \) for all \( x \in \mathbb{R} \).