1. Let $A$ be a real symmetric $n \times n$–matrix with $n$ distinct eigenvalues. Let $B$ be another $n \times n$–matrix, which commutes with $A$, i.e. $AB = BA$ following the matrix multiplication. Prove that $B = F(A)$, where $F(x)$ represents a polynomial in $x$.

2. (a) For a matrix $A$, the exponential function of the matrix $A$ is a matrix defined formally by the infinite series:

$$exp(A) = e^A = I + \sum_{1}^{\infty} \frac{A^n}{n!}.$$ 

Prove that $e^A$ is always nonsingular.

(b) Let $A$ be a $n \times n$–matrix defined as follows: $a_{k,k} = 2, k = 1, 2, \cdots, n; a_{1,n} = 1; a_{k-1,k} = 1, k = 2, 3, \cdots n; a_{j,k} = 0$(the remaining entries are zero.)

$$
\begin{pmatrix}
2 & 0 & 0 & \cdots & 0 & 1 \\
1 & 2 & 0 & \cdots & 0 & 0 \\
0 & 1 & 2 & \cdots & 0 & 0 \\
: & : & : & \cdots & : & : \\
0 & \cdots & 0 & 1 & 2 & 0 \\
0 & 0 & \cdots & 0 & 1 & 2
\end{pmatrix}
$$

Calculate determinant of $A$. (The principal diagonal consists of 2’s and the diagonal immediately below has 1. The right-most entry in the first row is also 1.)

3. Let $V$ denote a finite-dimensional linear vector space with an inner product (scalar product) and let $T : V \rightarrow V$ be a linear operator. Let $T^*$ denote the adjoint of $T$ defined by the relation $(Tv_1, v_2) = (v_1, T^*v_2)$. Prove the following:

$$R(T^*) = N(T)^\perp, N(T) = R(T^*)^\perp.$$