Applied Mathematics and Statistics
Common Qualifying Examination Part B
in Computational Applied Mathematics

Summer 2013 (May)

(Closed Book Exam)

Please solve 3 out of 4 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part B: 1 2 3 4

NAME ________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: May 27th, 2012
Time: 11:00 AM – 12:00 PM
B1. Classify the singularity of the differential equation

\[ x^4 y'' - x^2 y' + \frac{1}{4} y = 0. \]

at \( x = 0 \). Obtain the leading behavior of two solutions of this equation as \( x \to 0^+ \).
B2. Four ants, initially at rest at the four corners of a unit square centered at the origin, start walking counter-clockwise, each ant walking directly toward the instantaneous position of the one in front of him. Suppose each ant walks with unit velocity.

a) Find a differential equation for the trajectory of one ant in the Cartesian coordinate system

b) Solve this differential equation by making a transformation to the polar coordinate system. Sketch the solution.
B3. Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, and $B \in \mathbb{R}^{n \times n}$ is symmetric.

a) (4 points) Show that matrix $AB$ is diagonalizable, and all its eigenvalues are real.

b) (3 points) Show that matrices $AB$ and $BA$ have the same set of eigenvalues.

c) (3 points) Are the eigenvectors of $AB$ orthogonal to each other? Why or why not?
B4. Assume $A \in \mathbb{R}^{m \times n}$ has full rank, where $m \geq n$. Suppose $A$ is composed of a subset of rows of $B$. Let $A = Q_1 R_1$ and $B = Q_2 R_2$ be the reduced QR factorizations of $A$ and $B$, respectively.

a) (4 points) Show that $\|A\|_p \leq \|B\|_p$ for any $p \in [1, \infty)$.

b) (2 points) Show that $\|R_1\|_2 \leq \|R_2\|_2$.

c) (4 points) Show that $\|R_1^{-1}\|_2 \geq \|R_2^{-1}\|_2$. 