Qualitative Finance Qualifying Exam
2013 Summer

INSTRUCTIONS
You have 4 hours to do this exam.

Reminder: This exam is closed notes and closed books. No electronic devices are permitted. Phones must be turned completely off for the duration of the exam.

PART 1: Do 2 out of problems 1, 2, 3.
PART 2: Do 2 out of problems 4, 5, 6.
PART 3: Do 2 out of problems 7, 8, 9.
PART 4: Do 2 out of problems 10, 11, 12.

All problems are weighted equally.
On this cover page write which eight problems you want graded.

Problems to be graded:

______________________________________________
Academic integrity is expected of all students at all times,
whether in the presence or absence of members of the faculty.
Understanding this, I declare that I shall not give, use,
or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

______________________________________________
Signature

Stony Brook University
Applied Mathematics and Statistics
Problem 1
State the definition of martingale, and prove or disprove: \( \{ t - W^2(t) \} \) is a martingale with respect to the natural filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) generated by Brownian motion \( \{ W(t) \}_{t \geq 0} \).

**In order to receive full credit**, please apply **both** methods:

(i) By definition, compute the conditional expectation, for any \( 0 \leq s \leq t \),

\[
\mathbb{E}[t - W^2(t) | \mathcal{F}_s];
\]

(ii) Use Itô’s lemma to find

\[
d(t - W^2(t)).
\]
Problem 2

Suppose the market has a risk-free asset with return \( r_f \), and \( n \) risky assets. When short selling is allowed, the minimum-variance portfolio with the expected return \( \mu_* \) can be computed by solving the optimization problem:

\[
\begin{align*}
\min_{\omega} \omega^T \Sigma \omega & \quad \text{subjected to} \quad \omega^T \mu + (1 - \omega^T 1)r_f = \mu_*, \\
\end{align*}
\]

where \( \mu \) and \( \Sigma \) are the mean and covariance matrix of returns of \( n \) assets, and \( 1 = (1, 1, ..., 1)^T \).

Show that the solution for \( \omega \), when \( \Sigma \) is nonsingular and \( \mu \neq r_f 1 \), is

\[
\omega_{\text{eff}} = \frac{\mu_* - r_f}{(\mu - r_f 1)^T \Sigma^{-1} (\mu - r_f 1)} \Sigma^{-1} (\mu - r_f 1).
\]
Problem 3

Consider the one period-binomial pricing model for European call options. The initial price of the underlying stock is $S_0 = 20$, and the price could be either up to $S_1^{(1)} = 25$, or down to $S_1^{(2)} = 16$ at the maturity $T = 1$. The strike price $K = 22$, and the risk-free interest rate $r_f = 1/6$. Transaction cost is not considered here.

(i) Is this a non-arbitrage model?
(ii) What is the risk-neutral price of this call option?
(iii) If the market price turns out to be 3, then try to build an arbitrage by specifying your portfolio.
Problem 4

Consider a bond portfolio containing $N$ bonds with weights $x_i, i = 1, \ldots, N$ and one liability. The bond prices are given by

$$P_i = \sum_{t=1}^{T} F_{it} e^{-r_t t}, \quad i = 1, \ldots, N,$$

where $F_{it}$ is the future payoff of bond $i$ and $r_t$ is the riskless rate at time $t$. The present value of the liability is $P_L$. Changes in the term structure of interest rates are expressed as a linear combination of $k$ independent factors:

$$\Delta r_t = \sum_{j=1}^{k} \beta_{jt} \Delta f_j + \epsilon_t,$$

where $\epsilon_t$ is the error term assumed to be normally distributed with mean 0. For small but not necessarily parallel-shifts of the term structure, give the first-order condition for factor immunization.
Problem 5

Assume that a non-dividend-paying stock has an expected return of $\mu$ and a volatility $\sigma$. The stock price follows a geometric Brownian motion. An innovative financial institution has announced that it will trade a security that pays off a dollar amount equal to $\ln S_T$ at time $T$, where $S_T$ denotes the value of the stock price at time $T$.

(i) Use risk-neutral valuation to calculate the price of the security at time $t$ in terms of the stock price $S$ at time $t$, for $t < T$. $r$ is a constant representing the risk-free rate.

(ii) Confirm that your price satisfies the Black-Scholes partial differential equation.
Problem 6

The stock return process is defined by a stochastic differential equation,

\[ \frac{dS}{S} = (\mu X + \theta Y)dt + \sigma \sqrt{Y} dW_1, \]

where \( \mu, \theta, \) and \( \sigma \) are positive constants, \( X \) and \( Y \) are state variables, and \( W_1 \) is a scalar standard Brownian motion. The dynamics of the state variable \( X \) and \( Y \) are given by

\[ dX = (a - bX)dt + c\sqrt{X}dW_2 \]
\[ dY = (d - eY)dt + f\sqrt{Y}dW_3 \]

where \( a, b, c, d, e, f > 0 \), and \( W_2 \) and \( W_3 \) are scalar standard Brownian motions. Since changes in \( X \) are assumed to represent changes that are unrelated to the uncertainty, we require that \( W_2 \) be uncorrelated with \( W_1 \) and \( W_3 \). We require \( \theta > \sigma^2 \), which guarantees that the riskless rate is non-negative. \( W_1 \) and \( W_3 \) are correlated with \( dW_1dW_3 = \rho dt \), where \( -1 \leq \rho \leq 1 \).

Using Monte Carlo simulation, generate \( N \) scenarios of stock price from time \( t \) to time \( T \), \( 0 < t < T \).
Problem 7

Consider the stochastic differential equation

\[ dX(u) = (a(u) + b(u)X(u)) \, du + (\gamma(u) + \sigma(u)X(u)) \, dW(t) \]

where \( W(u) \) is Brownian motion relative to a filtration \( \mathcal{F}(u) \), and \( a(u), b(u), c(u) \) and \( d(u) \) are processes adapted to this filtration. Fix an initial time \( t \geq 0 \) and an initial position \( x \in \mathbb{R} \). Define

\[
Z(u) = \exp \left\{ \int_t^u \sigma(v) \, dW(v) + \int_t^u \left( b(v) - \frac{1}{2} \sigma^2(v) \right) \, dv \right\},
\]

\[
Y(u) = x + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)} \, dv + \int_t^u \frac{\gamma(v)}{Z(v)} \, dW(v).
\]

Show that \( X(u) = Y(u)Z(u) \) satisfies the stochastic differential equation such that \( X(t) = x \).
Problem 8
Let \{\Omega_2, \mathcal{F}_2, \mathbb{P}\} be such that \Omega_2 is the outcome space of two independent coin tosses, \mathcal{F}_2 is the \sigma\text{-algebra generated by all possible outcomes, and the coin is fair. Let } S_0, S_1, S_2 \text{ be the stock price model given by } S_0 = 4, S_i = 2S_{i-1} \text{ if the } i\text{th toss is a head, and } S_i = S_{i-1}/2 \text{ if the } i\text{th toss is a tail, } i = 1, 2. \text{ Compute the generalized conditional expectation } \mathbb{E}(S_2|\mathcal{G}), \text{ where } \mathcal{G} = \{\{HH, HT\}, \{TH, TT\}, \emptyset, \Omega_2\}. 
Problem 9
An American put with infinite expiry is called a *perpetual American put*. Consider the perpetual American put under the exercise rule: when the asset value $S(t)$ falls to level $L$ exercise the option. Under this exercise rule for fixed arbitrary $L > 0$, the risk neutral value $v_L(x)$ of the option is given by:

$$v_L(x) = \begin{cases} 
K - x, & \text{if } 0 \leq x \leq L, \\
(K - L) \left( \frac{x}{L} \right)^{-\frac{2r}{\sigma^2}}, & \text{if } x \geq L. 
\end{cases}$$

Derive the optimal exercise value $L^*$. 
**Problem 10**

Suppose that the change of the asset value $X$, over one period, follows the standard $t$-distribution with $\nu$ degrees of freedom. Compute the $100(1 - \alpha)\%$ expected shortfall (ES) of a long position over one period. The density of a standard $t$-distribution with $\nu$ degrees of freedom is

$$
\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}.
$$
Problem 11

Consider the following AR(1)-GARCH(1,1) model for daily return $r_t$,

$$r_t = \theta r_{t-1} + u_t, \quad u_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2,$$

where $-1 < \theta < 1$, $\theta \neq 0$, $\omega > 0$, $\alpha, \beta \in (0, 1)$ and $\alpha + \beta < 1$. What is the 99% 2-day VaR of a long position at time $t$?
Problem 12
Let $\mathbf{X} = (X_1, X_2)^T$ be a bivariate Gaussian copula with correlation $\rho$ and continuous margins. Show that the Kendall’s $\tau$ is

$$\rho_\tau(X_1, X_2) = \frac{2}{\pi} \arcsin \rho.$$
Scratch Paper 1
Scratch Paper 3
Scratch Paper 6
Scratch Paper 7