1. Suppose that $Y|Z = z \sim N(\frac{\alpha}{z}, \beta)$ and that $Z \sim Gamma(\frac{\alpha}{2}, \beta)$, where $\alpha, \beta > 0$.
   (a) Find the marginal pdf of $Y$.
   (b) Find the marginal variance of $Y$.
   (c) Suppose that $T \sim t_\alpha$. Find a function $g$ such that if $W = g(\mu, \alpha, \beta, T)$, then $W$ and $Y$ have the same distribution. Note that $\mu, \alpha$ and $\beta$ are constant.

Remark: the density of a $t$ distribution with $\nu$ d.f.is:
$$f(x|\nu) = \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1 + \frac{x^2}{\nu}\right)^{(\nu+1)/2}}$$

2. Suppose we have two independent random samples from two normal populations, that is, $X_1, X_2, \ldots, X_n \sim N(\mu_1, \sigma_1^2)$, and $Y_1, Y_2, \ldots, Y_n \sim N(\mu_2, \sigma_2^2)$. Furthermore, we know that $\mu_1 = \mu = \mu_2/2$, $\sigma_1^2 = \sigma_2^2 = \sigma^2/4$, and both $\mu$ and $\sigma^2$ are unknown parameters unless otherwise specified.
   (a) Please derive the maximum likelihood estimators (MLE’s) for $\mu$ and $\sigma^2$.
   (b) Are the above MLE’s for $\mu$ and $\sigma^2$ unbiased estimators for $\mu$ and $\sigma^2$ respectively?
   (c) Please derive the method of moment estimators (MOME’s) for $\mu$ and $\sigma^2$.
   (d) Are the above MOME’s for $\mu$ and $\sigma^2$ unbiased estimators for $\mu$ and $\sigma^2$ respectively?
   (e) Is there a best estimator (UMVUE) for $\mu$? Is there a best estimator (UMVUE) for $\sigma^2$?
   (f) Now assuming that $\mu$ is known, is there a best estimator (UMVUE) for $\sigma^2$?

3. Suppose we have two independent random samples from two normal populations, that is, $X_1, X_2, \ldots, X_n \sim N(\mu_1, \sigma_1^2)$, and $Y_1, Y_2, \ldots, Y_n \sim N(\mu_2, \sigma_2^2)$. Furthermore, although both population variances are unknown, we do know that $\sigma_1^2/\sigma_2^2 = c$, where $c$ is a known constant. Please derive a $100(1 - \alpha)%$ confidence interval for $(3\mu_1 - 2\mu_2)$ using the pivotal quantity method. Please include the derivation of the pivotal quantity, the proof of its distribution, and the derivation of the confidence interval for full credit.

4. Let the random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_c)$ have a multinomial distribution with $m$ trials and cell probabilities $\mathbf{\pi} = (\pi_1, \pi_2, \ldots, \pi_c)$, and the covariance matrix of $\mathbf{Y}$, $COV(\mathbf{Y}) = \Sigma$.
   (a) Show that $\Sigma = m[diag(\mathbf{\pi}) - \mathbf{\pi}'\mathbf{\pi}]$. 

(b) Let $\Sigma_{c-1}$ be the $(c-1) \times (c-1)$ upper-left sub-matrix in $\Sigma$ and $\pi_{c-1}$ be the first $(c-1)$ elements in $\pi$. Show that $\Sigma^{-1} = [\text{diag}(\pi_{c-1})^{-1} + 1' / \pi_c] / m$ is the inverse of $\Sigma_{c-1}$.

(c) Suppose we want to test $H_0: \pi_j = \pi_{j0}$ for $j = 1, ..., c$ versus $H_1$: Not $H_0$. Show that under $H_0$, the test statistics $X^2 = \sum_{j=1}^{c} \frac{(y_j - m\pi_{j0})^2}{m\pi_{j0}}$ asymptotically follows a chi-squared distribution.

(d) For the same test in (c), find the likelihood ratio test statistics, denoted as $G^2$, and show that under $H_0$, it has the same asymptotic distribution as $X^2$.

Remark: you may use the following theorem without proof: Let $X$ be a multivariate normal with mean $\mu$ and covariance matrix $B$. A necessary and sufficient condition for $(X - \mu)'C(X - \mu)$ to have a chi-squared distribution is $BCBCB = BCB$. The degrees of freedom equal the rank of $CB$. 