Qualifying Exam (2012 Summer):
Quantitative Finance

You have 4 hours to do this exam.

**Reminder:** This exam is closed notes and closed books.
Do 2 out of problems 1,2,3.
Do 2 out of problems 4,5,6.
Do 3 out of problems 7,8,9,10,11,12.

All problems are weighted equally. **On this cover page**
write which seven problems you want graded.

**problems to be graded:**

Academic integrity is expected of all students at all times,
whether in the presence or absence of members of the faculty.
Understanding this, I declare that I shall not give, use,
or receive unauthorized aid in this examination.

**Name (PRINT CLEARLY), ID number**

**Signature**
Let \((N_t)_{t \geq 0}\) be Poisson process with parameter \(\lambda\) and let \(\{Y_1, Y_2, \cdots\}\) be an independent sequence of random variables with
\[
Y_j = \begin{cases} 
1 & \text{with probability } p \\
-1 & \text{with probability } 1 - p 
\end{cases}, \text{ for } j = 1, 2, \cdots.
\]
Consider a process \((X_t)_{t \geq 0}\) defined by
\[
X_t = \sum_{k=1}^{N_t} Y_k.
\]
Compute characteristic function \(\phi_{X_t}(u)\) of \(X_t\) and find a measure \(\nu(dx)\) such that
\[
\phi_{X_t}(u) = \exp \left( t \int_{\mathbb{R}} (e^{iux} - 1) \nu(dx) \right).
\]
Let \( B = (B_t)_{0 \leq t \leq T} \) be Brownian motion under measure \( \mathbb{P} \) and the time horizon \( T > 0 \), and let \( M \) be the maximum of \( B \) up to time \( t \): \( M_t = \max_{0 \leq s \leq t} B_s \).

(a) Drive the joint probability density function \( f_t(x, y) \) of \((B_t, M_t)\):

\[
f_t(x, y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} \mathbb{P}(B_t \leq x, M_t \leq y).
\]

using “Reflection Principal”.

(b) Let \( W_t = \mu t + B_t \), and \( W^*_t = \max_{0 \leq s \leq t} W_s \). Find probability density function

\[
g_t(x, y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} \mathbb{P}(W_t \leq x, W^*_t \leq y).
\]

**Theorem 1** (Reflection Principal). Let \( B = (B_t)_{t \geq 0} \) be Brownian motion and \( T \) be a stopping time. Define a process \( \hat{B} = (\hat{B}_t)_{t \geq 0} \) as

\[
\hat{B}_t = \begin{cases} 
B_t & \text{for } t \leq T \\
2B_T - B_t & \text{for } t \geq T
\end{cases}.
\]

Then \( \hat{B} \) is also Brownian motion.
(3). Let $B = (B_t)_{0 \leq t \leq T}$ and $N = (N_t)_{0 \leq t \leq T}$ be a Brownian motion and a Poisson process with parameter 1, respectively, under a measure $\mathbb{P}$ and time horizon $T > 0$, and let $X_t = B_t + N_t$. Find an equivalent measure $\mathbb{Q}$ of $\mathbb{P}$ such that $X = (X_t)_{0 \leq t \leq T}$ is martingale under $\mathbb{Q}$ and deduce that there are infinitely many equivalent martingale measure.

**Hint** Use Girsanov’s theorem for Poisson processes together with Girsanov’s theorem for Brownian motion.

**Theorem 2** (Girsanov’s theorem for Poisson processes). Let $N = (N_t)_{0 \leq t \leq T}$ be a Poisson process with parameter $\mu$ under measure $\mathbb{P}$. For a constant $\lambda > 0$, define $\mathbb{Q}$ by

$$
\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{(\mu - \lambda)T + N(T)(\log \lambda - \log \mu)}.
$$

Then $N$ is a Poisson process with parameter $\lambda$. 

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4
(4). Suppose the market is complete. The stock price process \((S_t)_{t \geq 0}\) is modeled by the Itô stochastic differential equation as

\[ dS_t = \mu S_t dt + \sigma S_t dB_t \]

where \((B_t)_{t \geq 0}\) is the standard Brownian Motion and \(\mu\) and \(\sigma\) are constant positive real numbers. The risk free money account price \((\beta_t)_{t \geq 0}\) is defined by

\[ d\beta_t = r \beta_t dt \]

with risk free rate of return \(r\). Suppose we have an European contingent claim underlying the stock price \(S_T\) at the maturity time \(T > 0\). That is the payoff of the contingent claim at the maturity time \(T\) is \(g(S_T)\). Let \((Y_t)_{0 \leq t \leq T}\) be the price process of the contingent claim with \(Y_T = g(S_T)\). Deduce the partial differential equation (Black-Scholes equation) for pricing the contingent claim under the no-arbitrage assumption.
(5). Under the Hull and White interest rate model, interest rate $r_t$ at time $t \geq 0$ is given by the following stochastic differential equation

$$dr_t = (b - ar_t)dt + \sigma(t)dW_t,$$

where $a$, and $b$ are positive constants, $\sigma(t)$ is a deterministic function, and $(W_t)_{t \geq 0}$ is Brownian Motion. Solve the equation and compute $\text{var}(r(t))$ and $\text{cov}(r_s, r_t)$ for $0 \leq s \leq t$. 
(6). The tempered stable process $(X_t)_{t \geq 0}$ is a Lévy process whose characteristic function is

$$
\phi_{X_t}(u) = E_Q[\exp(iuX_t)] = \exp(i\mu t - iut\Gamma(1 - \alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})
+ t\Gamma(-\alpha)((\lambda_+ - iu)\alpha - \lambda_+ + (\lambda_- + iu)^\alpha - \lambda_-)),
$$

where $\alpha \in (0, 2)$, $C, \lambda_+, \lambda_- > 0$, and $m \in \mathbb{R}$. Note that $\phi_{X_t}(u + i\rho)$ is defined for all $u \in \mathbb{R}$ if $\rho \in [-\lambda_+, \lambda_-]$.

Let $(X_t)_{t \geq 0}$ be the tempered stable process with parameters $(\alpha, \lambda_+, \lambda_-, C, m)$ and $\lambda_+ > 1$, under the risk neutral measure $Q$ and $E_Q[\exp(X_t)] = \exp(rt)$ with the risk free rate of return $r$ with $r < \lambda_+$. Suppose that the stock price process $S = (S_t)_{t \geq 0}$ is given by the exponential Lévy process model as follows:

$$S_t = S_0 \exp(X_t), \quad t \geq 0$$

Consider an European call option underlying $S$ with the strike $K$ and the maturity $T$. Calculate the option pricing formula for the European option at time $t$, $0 \leq t \leq T$.

**Hint:** Use the complex inversion formula.
(7). Assume that the daily index return process \((y_t)_{t=0,1,2,\ldots,N}\) follows GARCH(1,1) model:

\[
\begin{align*}
y_t &= \sigma_t \varepsilon_t + c \\
\sigma_t^2 &= \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2 
\end{align*}
\]

where \((\varepsilon_t)_{t=0,1,2,\ldots,N}\) is i.i.d and \(\varepsilon_t \sim N(0,1)\) for all \(t = 0, 1, 2, \ldots, N\).

(a) Explain the maximum likelihood method to estimate parameters \(\alpha_0, \alpha_1, \beta_1,\) and \(c\).

(b) Discuss forecasting the value at risk (VaR) of the index return for time \(t + 1\) under information until time \(t\).
(8). Let $\alpha \in (0, 2)$, $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$. The stable random variable $X$ with parameter $(\alpha, \sigma, \beta, \mu)$ is defined by the characteristic function

$$E[e^{iuX}] = \begin{cases} \exp\left(-\sigma|u|^\alpha(1 - i\beta\text{sign}(u) \tan \frac{\pi\alpha}{2}) + i\mu u\right), & \alpha \neq 1 \\ \exp\left(-\sigma|u|(1 + i\beta\frac{2}{\pi}\text{sign}(u) \ln |u|) + i\mu u\right), & \alpha = 1 \end{cases}$$

where

$$\text{sign}(u) = \begin{cases} 1, & u \geq 0 \\ 0, & u = 0 \\ -1, & u \leq 0 \end{cases}$$

and we denote $X \sim S_{\alpha}(\sigma, \beta, \mu)$.

Let $R = (R_1, R_2)^T$ be a bivariate random vector defined by $R = \sqrt{A}Z$ where $Z$ is a bivariate normal random vector with $Z = (Z_1, Z_2)^T \sim N(0, \Sigma)$, $A$ is an univariate stable random variable with $A \sim S_{\alpha/2}(\sigma, 1, 0)$ for $\alpha \in (0, 2)$ and $\sigma > 0$, and $A$ and $Z$ are independent. What is the distribution of a new random variable $R_p = w_1R_1 + w_2R_2$ where $w_1, w_2 > 0$?

**Hint:** The laplace transform of $A$ is given by

$$L_A(u) = E[\exp(-uA)] = \exp\left(-\frac{\sigma^{\alpha/2}}{\cos(\pi\alpha/4)} u^{\alpha/2}\right),$$

for $u > 0$. 


We would like to price a put option with the maturity $T$ and the strike price $K$ under the Gamma-subordinated geometric Brownian motion model. Let $(B_{1,t}, B_{2,t})_{t \geq 0}$ be the bivariate standard Brownian Motion with $dB_{1,t}dB_{2,t} = \rho dt$ for $\rho \in \mathbb{R}$. The risk-neutral stock price process $(S_t)_{t \geq 0}$ is modeled by the following dynamic:

$$
\begin{align*}
    dS_t &= rS_t dt + \sqrt{V_t}S_t dB_{1,t} \\
    dV_t &= (b - aV_t) dt + \lambda \sqrt{V_t} dB_{2,t}
\end{align*}
$$

where $r > 0$ is the risk-free rate, and $a > 0$, $b > 0$, and $\lambda > 0$, and $V_0 = b$. Describe Monte Carlo analysis to determine the price of the option at time 0 using $N$ trials.
Consider a portfolio consists of 3 stocks. The random variables $R_1$, $R_2$, and $R_3$ are for returns of assets in the portfolio. Assume that $\mu_1 = E[R_1] = 0.1$, $\mu_2 = E[R_2] = 0.2$, $\mu_3 = E[R_3] = 0.3$, $var(R_1) = var(R_2) = var(R_3) = 0.5$, and covariances $cov(R_n, R_m) = 0$ for all $m, n \in \{1, 2, 3\}$ with $m \neq n$. Let $\bar{\mu} = 0.1$ be the expected portfolio return. Find the optimal portfolio allocation weights $w_1$, $w_2$, and $w_3$ of $R_1$, $R_2$, and $R_3$, respectively, by solving the following problem:

$$\min_w \text{var} \left( \sum_{n=1}^{3} w_n R_n \right)$$

s.t.

$$\sum_{n=1}^{3} w_n \mu_n = \bar{\mu}$$

$$\sum_{n=1}^{3} w_n = 1.$$ 

Short selling is allowed.
Consider a portfolio consists of two stocks and let $R_1$, and $R_2$ be continuous random variables of returns of the two stocks. Let $F(x, y)$ be the joint distribution of $(R_1, R_2)$. Define copula and calculate probability $P[R_1 \leq -\text{VaR}_\alpha(R_1) \text{ and } R_2 \leq -\text{VaR}_\alpha(R_2)]$ using copula, where $\text{VaR}_\alpha(R_n)$ is the value at risk of $R_n$ at $\alpha$ significance level for $n = 1, 2$. 

(11).
(12). Under Merton’s firm value model, dynamic of a firm value is given by the following equation under risk neutral measure:

\[ dV_t = rV_t dt + \sigma V_t dW_t, \]

where \( r \) is the risk free rate of return and \( (W_t)_{t \geq 0} \) is Brownian Motion. Let \( \bar{B}(t, T) \) be the price of a defaultable zero coupon bond at time \( t \) with the maturity \( T \) and the principal \( D \). The defaultable bond payoff at time \( T \) is given by

\[ \bar{B}(T, T) = \begin{cases} D, & V_T > D \\ V_T, & V_T \leq D \end{cases} \]

(a) What is the risk neutral price \( \bar{B}(0, T) \) at time \( t = 0 \)?
(b) What is the probability that the firm will default in time interval \( (0, T) \)?