APPLIED MATHEMATICS and STATISTICS
DOCTORAL QUALIFYING EXAMINATION
in COMPUTATIONAL APPLIED MATHEMATICS

Summer 2009 (June)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A: 1 2 3 4 5
Part B: 6 7 8 9 10

NAME ________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: June 3rd, 2009
Time: 9:00 – 1:00 PM
A1. Construct Green’s function to solve the boundary value problem

\[ \frac{d^2y}{dx^2} + 4y = f(x), \quad y(0) = 0, \quad y(\pi/4) = 0. \]

Now consider this equation on the interval \([0, L]\) with the same boundary condition \(y(0) = 0, \quad y(L) = 0\).
For what values of \(L\) is this problem impossible? Explain why those values are special.
A2. Solve exactly the following initial value problem

\[ y' + y + \varepsilon y^2 = 0, \quad y(0) = 1. \]

Analyze the limiting behavior of the solution as \( \varepsilon \to 0 \) for \( x \leq 0 \) and \( x \geq 0 \). Should the problem be classified as regular or singular (in the sense of perturbation theory) for \( x \) positive? \( x \) negative?
A3. Consider the initial boundary value problem

\[ u_{tt} - c^2 u_{xx} = 0, \quad x, t > 0. \]
\[ u(x, 0) = g(x), \quad x > 0, \]
\[ u_t(x, 0) = h(x), \quad x > 0, \]
\[ u(0, t) = 0, \quad t \geq 0. \]

with \( g(0) = 0 = h(0) \). If we extend \( g, h \) as odd functions on \( -\infty < x < \infty \), show that d’Alembert’s formula gives the solution.
A4. Prove the theorem:

\( f(z) \) is analytic in \( N_R(z_0) \), the followings are equivalent:

1. \( z_0 \) is a pole.
2. \( f(z) = \frac{g(z)}{(z - z_0)^m} \), where \( m \) is a positive integer, \( g(z) \) is analytic and \( g(z) \neq 0 \) in \( N_\delta(z_0), (\delta \leq R) \).
3. The principle part of the Laurent expansion of \( f(z) \) has a finite number of terms.
A5. Evaluate

(a) \[ \int_{0}^{2\pi} \frac{1}{2 + \cos \theta} d\theta. \]

(b) \[ \int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^2} dx. \]
B6. Let \( v \) be a nonzero vector in \( \mathbb{C}^m \).

a) What is the null space of \( I - \frac{vv^*}{v^*v} \)? What is the geometric interpretation of the transformations defined by this matrix (i.e., left-multiply a vector by this matrix)?

b) What is the null space of \( I - \frac{2vv^*}{v^*v} \)? What is the geometric interpretation of the transformations defined by this matrix?

c) Describe a procedure using pseudocode to evaluate \( (I - \frac{2vv^*}{v^*v})x \) as efficient as possible in terms of the number of flops. What is the number of flops of your procedure?
B7. Let $A = \begin{bmatrix} I & B \\ B^T & I \end{bmatrix}$, where $B \in \mathbb{R}^{m \times m}$ with $\|B\|_2 < 1$.

a) Show that the eigenvectors of $A$ are the columns of the matrix

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} U & U \\ V & -V \end{bmatrix},$$

where $B = U\Sigma V^T$ is a singular value decomposition of $B$.

b) Show that the condition number of $A$ in 2-norm is

$$\kappa(A) = \frac{1 + \|B\|_2}{1 - \|B\|_2}.$$
B8. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{1}{2}(x_1^2 - x_2)^2 + \frac{1}{2}(1 - x_1)^2.$$  

a) Use the first- and second-order optimality conditions to show that $x^* = [1, 1]$ is a local minimum.

b) Give the linear system for the first iteration of Newton’s method for minimizing $f$ using $x_0 = [2, 2]^T$ as starting point.

c) Give the first iteration of steepest descent method for minimizing $f$ using $x_0 = [2, 2]^T$ as starting point.
B9. Suppose $Q(f) = \sum_{i=1}^{n} w_i f(x_i)$ is a quadrature rule based on polynomial interpolation over the interval $[a, b]$.

a) What is the value of $\sum_{i=1}^{n} w_i$? Justify your answer.

b) Suppose that Lagrange interpolation at a given set of nodes $x_1, \ldots, x_n$ is used to derive the quadrature rule. Prove that the corresponding weights are given by the integrals of Lagrange basic functions, $w_i = \int_{a}^{b} \ell_i(x) \, dx, i = 1, \ldots, n$. 
B10. Suppose you are given a general-purpose subroutine for solving initial value problems for systems of first-order ODEs \( y' = f(t, y) \), and the subroutine supports integration using Euler’s method, backward Euler’s method, trapezoid method, and fourth-order Runge-Kutta method.

a) Discretize the following heat equation in space using centered-difference approximation for \( u_{xx} \) to obtain an initial value problem with a system of ODEs that is suitable for the general-purpose ODE solver.

\[
  u_t = cu_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0,
\]

with initial condition

\[
  u(0, x) = f(x), \quad 0 \leq x \leq 1,
\]

and boundary conditions

\[
  u(t, 0) = 0, \quad u(t, 1) = 0, \quad t \geq 0.
\]

b) Among the methods supported by the software, which methods are appropriate to solving the resulting system of ODEs?