Doctoral Qualifying Examination
Spring 2000

Advanced Calculus & Linear Algebra

Name: ____________________________
ID #: ____________________________

Start your answer on each question sheet. Attach all extra sheets you may have used to the appropriate sheet. Hand in all question sheets.

Time: 9:00AM-11:00AM
Date: January 19, 2000
Place: SBS N101
1. Let \( f(x) \) be a real-valued differentiable function with \( f'(x) = 0 \) for all real values of \( x \). Prove that \( f(x) \) is a constant.
2. Prove that for $\sigma > 0$,

$$\frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma}\right)^2\right) dx = 1.$$
3. Prove that for all values of $x > 0$,

$$\left(1 + \frac{1}{x}\right)^x < 2.9.$$
4. Consider the matrix mapping \( A : \mathbb{R}^4 \to \mathbb{R}^3 \) where

\[
A = \begin{pmatrix}
1 & 2 & 0 & 1 \\
2 & -1 & 2 & -1 \\
1 & -3 & 2 & -2 \\
\end{pmatrix}
\]

Find a basis and the dimension of (a) the image of \( A \) and (b) the kernel of \( A \).
5. Let $A$ be an $n \times n$ real symmetric matrix. Prove that all its eigenvalues are real and the eigenvectors corresponding to different eigenvalues are orthogonal.
6. For the quadratic form $Q(x, y, z) = 3x^2 + 2xy + 3y^2 + 2xz + 2yz + 3z^2$

(a) Find a symmetric matrix $A$ such that $(Au, u)$ is equal to $Q(x, y, z)$, where $u = (x, y, z)^T$.

(b) Using an orthogonal change of coordinates diagonalize $Q(x, y, z)$. 