DOCTORAL QUALIFYING EXAMINATION
FALL 2002
Advanced Calculus & Linear Algebra

NAME

SOLAR ID#

Start your answer on each question sheet. Attach all extra sheets you use to the appropriate sheet. Hand in all question sheets.

Note: Answer all questions.

Date: September 3, 2002
Time of Exam: 9AM-11AM
Place of Exam: Stony Brook Union Auditorium
1. Given:

\[ \int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2} \]

Derive the value of

\[ \int_0^\infty \frac{\sin^2 x}{x^2} \, dx \]
Let \( f(x) = 3x^2 \) for \( 0 \leq x \leq 1 \) and \( f(x) = 4 - x \) for \( 1 \leq x \leq 4 \). Let \( R \) be the region bounded by the \( x \)-axis, the graph of \( f \) and the straight line segments \( x = b \) and \( x = b + 2 \) connecting the graph to the \( x \)-axis. Find the value of \( b \) for which the area of \( R \) is maximum.
Prove by mathematical induction the identity

\[ \sum_{k=0}^{n} C(n, k) = 2^n. \]

Here \( C(n, k) \) denote the binomial coefficients \( \binom{n}{k} \).
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation $T(v) = Av$, where

$$A = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}$$

Find $\ker(T)$. 
Let $A$ be a skew-symmetric $n \times n$-real matrix, i.e., $A^T = -A$.

(a) Show that $A$ is singular if $n$ is odd.

(b) Show that all the eigenvalues of $A$ are purely imaginary.
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an affine transformation (a mapping which maps straight line segments into straight line segments) defined by the relation $T(u) = Au + b$, where $b$ is a specified vector in the cartesian plane and $A$ is a $2 \times 2$ matrix:

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

A vector $v$ is called a **fixed point** of $T$ if $Tv = v$. Show that $T$ has a unique fixed point if $(p - 1)(s - 1) - qr \neq 0$. 