Qualifying Exam (January 2010): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.
Do 2 out of problems 1,2,3.
Do 2 out of problems 4,5,6.
Do 3 out of problems 7,8,9,10.

All problems are weighted equally. **On this cover page write which seven problems you want graded.**

problems to be graded:

________________________________________________________

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Name (PRINT CLEARLY), ID number**

________________________________________________________

Signature

________________________________________________________
1). A Company manufactures desks, tables and chairs. Each item requires lumber, finishing hours and carpentry hours, according to the table below: We have 48 units of lumber, 20 hours of finishing, and 8 hours of carpentry.

<table>
<thead>
<tr>
<th></th>
<th>desk</th>
<th>table</th>
<th>chair</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lumber</td>
<td>8</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Finishing hours</td>
<td>4</td>
<td>2</td>
<td>1.5</td>
</tr>
<tr>
<td>Carpentry hours</td>
<td>2</td>
<td>1.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Defining \( x_i \) to be the number of items of type \( i \) made, we get the following LP:

\[
\begin{align*}
\text{max} \quad z &= 60x_1 + 30x_2 + 20x_3 \\
8x_1 + 6x_2 + x_3 &\leq 48 \\
4x_1 + 2x_2 + 1.5x_3 &\leq 20 \\
2x_1 + 1.5x_2 + 0.5x_3 &\leq 100 \\
x_1, x_2, x_3 &\geq 0
\end{align*}
\]

Let the slack variables be denoted \( s_i \). The optimal tableau was obtained:

<table>
<thead>
<tr>
<th></th>
<th>( z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>280</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-8</td>
<td>24</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>-4</td>
<td>8</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1.25</td>
<td>0</td>
<td>0</td>
<td>-0.5</td>
<td>1.5</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Each of the following parts is independent of the others:

(a). Find the dual of this LP and its optimal solution (the objective value and the value of the dual variables). Use the tableau – do not solve from scratch!

(b). A new constraint is added: \( x_2 \geq 1 \). Find the new optimal solution using sensitivity analysis.

(c). Find the range of values of the objective function coefficient for \( x_3 \) for which the current basis remains optimal.

(d). Find the range of values of the RHS coefficient \( b_1 \) (amount of lumber) for which the current basis remains optimal.

(e). A new item is suggested which sells for $50, requires 1 unit of lumber, 3 finishing hours, and 1.5 carpentry hours. Find the new optimal solution, using sensitivity analysis (do not solve from scratch!).

2). Let \( x^* \) be an optimal solution to the problem: Minimize \( cx \) subject to \( a^i x = b_i \) for \( i = 1, 2, ..., m \) and \( x \geq 0 \). ( \( a^i \) is the \( i \)th row of a matrix \( A \).) Let \( w^* \) be an optimal dual solution. Show that \( x^* \) is also optimal to the problem: Minimize \( (c - w^*_k a^k) x \) subject to \( a^i x = b_i \), for \( i = 1, 2, ..., m, i \neq k \) and \( x \geq 0 \), where \( w^*_k \) is the \( k \)th component of \( w \).

3). A company needs to process 9 jobs and has 3 identical machines on which do the work. Each job will be processed on a single machine, and all processing times are 1 hour. A machine can process one job at a time. Each job \( j = 1, ..., 9 \) has a deferral cost \( c_j(t) \) which is the cost to complete job \( j \) at time \( t \). We assume that these functions \( c_j(t) \) are increasing functions, so we want to complete jobs as soon as possible. (Note that the completion cost does not depend on the machine that processes the job.) We wish to find an assignment of jobs to machines, and a schedule for the jobs, completed at times \( t_1, ..., t_9 \) that minimizes the total deferral cost \( \sum_{j=1}^{9} c(t_j) \).
Formulate a Balanced Transportation Problem to solve this scheduling problem. Give your formulation in terms of a cost and requirement table, or by drawing an appropriate graph. Recall that a Transportation Problem is Balanced if the sum of the supplies is equal to the sum of the demands.

4). Cars arrive at a parking lot according to a Poisson process with rate $\lambda$. There are only four parking spaces, and any car that arrives when all the spaces are occupied goes elsewhere. The parking duration of a car is exponentially distributed with mean $1/\mu$. Find the average number of empty parking spaces in the steady state.

5). Consider an $M/G/1$ queueing system where service is rendered in the following manner. Before a customer is served, a biased coin whose probability of heads is $p$ is flipped. If it comes up heads, the service time is exponentially distributed with parameter $\mu$. If it comes up tails, the service time is equal to the constant $d$. The arrival rate is $\lambda$. Calculate the average number of customers in this system in the steady state.

6). A component is either replaced upon failure or upon reaching age $T > 0$. The lifetimes of successive components are independent and identically distributed random variables with probability density function $f(x)$. A cost $c_1$ is incurred for each planned replacement (at age $T$), and a cost $c_2 > c_1$ is incurred for each failure replacement. What is the long-run average cost per unit time?

7). Consider a random variable $X$ with the following probability density function

$$f(x) = \begin{cases} 
0.5 & -k < x < 0 \\
x & 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}$$

(a) Find the constant $k$.

(b) Give the inverse-transform, composition, and an acceptance-rejection algorithm for generating $X$. What is the efficiency of your acceptance-rejection algorithm? Discuss which of the three algorithms is preferable.

8). For each of the following distributions, derive formulas for the Maximum Likelihood Estimators (MLEs) of the indicated parameters. Assume that we have i.i.d. data $X_1, X_2, \ldots, X_n$ from the distribution in question.

1. Uniform distribution $U(a, b)$, joint MLEs for $a$ and $b$

2. Normal distribution $N(\mu, \sigma^2)$, joint MLEs for $\mu$ and $\sigma$

3. Binomial distribution $Bin(t, p)$, MLE for $p$ assuming that $t$ is known.

9). The price of a certain stock is fluctuating among $10$, $20$, and $30$ from month to month. Market analysis indicates that given that the stock is at $10$ in the current month, then in the following month it will be at $10$ with probability $4/5$ and at $20$ with probability $1/5$. Similarly, given that the stock is at $20$ in the current month, then in the following month it will be at $10$ with probability $1/4$, at $20$ with probability $1/2$, and at $30$ with probability $1/4$. Finally, given that the stock is at $30$ in the current month, then in the following month it will be at $20$ with probability $5/8$ and at $30$ with probability $3/8$. Given a discount factor of 0.9, use the policy iteration method to determine when to sell and when to hold the stock to maximize the expected total discounted profit over the infinite time horizon.

10). Consider an MDP with a finite state set $X$, compact action sets $A(x)$, $x \in X$, continuous in $a \in A(x)$ transition probabilities $p(y|x,a)$, and continuous in $a \in A(x)$ reward functions $r(x,a)$. Provide an example of such an MDP for which stationary optimal policies do not exist for the average reward per unit time criterion.