Qualifying Exam (Winter 2006): Operations Research

You have 4 hours to do this exam. Reminder: This exam is closed notes and closed books.
Do 2 out of problems 1,2,3.
Do 2 out of problems 4,5,6.
Do 3 out of problems 7,8,9,10,11,12,13,14.

All problems are weighted equally. On this cover page write which seven problems you want graded.

problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

Signature
1. Prove: EITHER: \( x \geq 0, Ax \leq b \) OR \( b \in \mathbb{R}^n, a \in \mathbb{R}^m \), and \( a^T x \leq b \), but not both. \( A \) is an arbitrary \((m \times n)\) matrix, and \( b \) is an \( n \)-vector. Note: It is easy to show that \( a \) and \( b \) specified cannot both exist. You must also show that if \( a \) does not exist then \( b \) must exist (or vice versa).

2. Consider the following optimal tableau for a maximization problem where the constraints are of the \( \leq \) type, and \( x_1, x_2, x_3, x_4 \) are the slack variables of the constraints. \( x_1, x_2, x_3, x_4 \) are the original variables.

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(a) Find the optimal objective value \( \theta \).
(b) What is the allowable increase and decrease of \( b_1 \) (the right hand side of the first constraint) that keeps the optimal tableau feasible?
(c) What is the optimal tableau if a new activity \( x_6 \) is added with coefficients \((2, 0, 3)^T\) in the constraints and price of 5 were added to the problem?
(d) Ignoring part (c), suppose that we add the constraint \( x_1 - x_2 + 2x_3 \leq 10 \) to the problem. What is the new optimal tableau?

3. (a) Prove the following constructively: For every basic feasible solution \( x \), there exists a vector \( c \) such that \( x \) is optimal to the LP \( \min \{ cx \mid Ax = b, x \geq 0 \} \). ("Prove constructively" means give the values of the components in such a vector \( c \)).
(b) In class we have shown several rules that prevent cycling when applying Simplex to an LP that may be degenerate. (Recall the lexicographic perturbation method and Bland’s method.) Prove that cycling can never occur even in the presence of degeneracy when applying the Simplex method without any cycling prevention rules, provided that the minimum ratio test always uses a unique solution. (In other words, show that if the minimum ratio test has a unique winner, Simplex will not cycle.)

4. Let \((S_k)_{k=1}^{\infty}\) be independent and identically distributed random variables with \( P(S_k = 1) = P(S_k = -1) = 0.5 \). Let \( X_0 = 0, S_n = X_1 + \cdots + X_n \), and \( M_n = \max\{S_k : 0 \leq k \leq n\} \). Does each of the following sequences:
(a) \((S_n)_{n=2}^{\infty}\)
(b) \((M_n)_{n=2}^{\infty}\)
(c) \(M_n - S_n\) form a Markov chain? Explain your answers.

5. Consider a single-server Markovian queue with unlimited waiting space. The service intensity is \( \mu \) and the arrival rate is \( \lambda \). If an arrival sees \( n \) customers in the system, the customer joins the queue with probability \( \frac{1}{n+1} \). Find the average amount of time that an arriving customer spends in the system.

6. Consider a single-server bank in which customers arrive in accordance with a Poisson process with rate \( \lambda \). A customer will enter the bank only if there are no other customers there. The customer service times are independent and identically distributed random variables with the distribution function \( F \). Find the probability that an arriving customer enters the bank. Explain your answer.

7. Stony Brook University wishes to assign professors to courses for the next academic year. Suppose that there are a total of 21 professors and 4 different courses to be offered. For course \( j \), at least \( a_j \) sections
should be offered to accommodate the demand, for \( j = 1, \ldots, n \). For each professor \( i \), let \( C_i \) denote the set of courses she can teach, \( i = 1, \ldots, a \). Each professor can teach at most \( c \) courses in any academic year. Furthermore, a professor cannot teach more than \( e \) sections of the same course. If a professor is not assigned any courses to teach, then she will be on leave (without pay) for the entire academic year. Let \( P_i \) denote the annual salary of professor \( i, i = 1, \ldots, a \). Formulate an integer programming problem to find an assignment of the courses to professors so as to minimize the total salary paid during the next academic year.

8) Consider the generic nonlinear optimization problem

\[
\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,
\]

where \( f: \mathbb{R}^n \to \mathbb{R}, g: \mathbb{R}^m \to \mathbb{R}^m, \) and \( h: \mathbb{R}^p \to \mathbb{R}^p \). Suppose that the Mangasarian-Fromovitz constraint qualification holds at a feasible \( x \), i.e., \( \nabla h(x) \) has full column rank and there exists \( d \) such that \( \nabla g(x)^T d < 0 \) and \( \nabla h(x)^T d = 0 \). Suppose that \( x \) is a local minimizer of \( P \). Let \((u, v)\) be the corresponding Lagrange multipliers. Show that the set of vectors \((u, v)\) satisfying the KKT conditions is bounded, i.e., there exists a finite number \( M > 0 \) such that \( \|u\| \leq M, \|v\| \leq M \) for all Lagrange multipliers \((u, v)\).

9) Consider the CDF

\[
F(x) = \begin{cases} 
0 & x < 0 \\
(2-a)x - (1-a)x^2 & 0 \leq x \leq 1 \\
1 & x > 1
\end{cases}
\]

where \( a \) is a parameter, with \( 0 < a < 1 \). Give two different methods by which we can generate independent samples from the distribution. For each method, be as detailed and precise as you can: your answers should come as close to providing a detailed, specific algorithm for this particular distribution as you can, rather than a general description of the method.

10) Suppose we want to simulate the following random walk problem. A gambler has \( x \) dollars. In each play, the probability of winning \( 1 \) dollar is \( p \), the probability of losing \( 1 \) dollar is \( q = 1 - p \). The gambler will stop either when he reaches \( b > x \) dollars or when he reaches \( a < 0 \) dollars.

(a) Suppose the probability for the gambler to lose (stays at \( a \)) is \( \phi(x) \). What is the relation between \( \phi(x) \) and \( \phi(x-1), \phi(x+1) \)?

(b) What is the formula for \( \phi(x) \)?

(c) Describe the algorithm for simulating the gambler in details.

11) We want to solve the following query problem: Given a set \( P \) of \( n \) disjoint line segments in the plane, determine if a query line \( l \) intersects all \( n \) of the segments (in which case we call it a "stubber"). We want to preprocess the input data so that we can answer queries very fast. The output to a query will be "yes" if the line \( l \) intersects all \( n \) segments and will be "no" elsewhere.

(a) Describe briefly a method for solving this problem. Try to be as efficient as possible in both space and query time.

(i). Preprocessing time is \( O(\ ) \)

(ii). Storage space (memory usage) is \( O(\ ) \)

(iii). Query time is \( O(\ ) \)

(b) Assume now that we know that \( l \) is always axis-parallel (horizontal or vertical). How efficiently can you now solve the problem? Give a brief justification.

(i). Preprocessing time is \( O(\ ) \)

(ii). Storage space (memory usage) is \( O(\ ) \)

(iii). Query time is \( O(\ ) \)
12. Let \( S = \{ \text{the set of points in the plane} \} \).

(a) What is the definition of a point in \( S \)?

(b) How efficiently can we determine a point in \( S \)?

(c) How efficiently can we determine the point that minimizes the Euclidean distance between two points in \( S \)?

13. Let \( X \) and \( Y \) be independent random variables distributed as \( X \) and \( Y \), respectively, in probability \( P \). Suppose that \( X + Y \) is independent of \( X \) and \( Y \) in probability \( P \). Assume that \( X \) and \( Y \) are integers in \([0, N]\) with \( N \) large. What is the probability that \( X + Y \) is divisible by 3?