Qualifying Exam (January 2007): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.
Do 2 out of problems 1,2,3.
Do 2 out of problems 4,5,6.
Do 3 out of problems 7,8,9,10,11,12,13,14,15,16.

All problems are weighted equally. On this cover page write which seven problems you want graded.

**problems to be graded:**

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Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Name (PRINT CLEARLY), ID number**

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**Signature**

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1). A company manufactures three types of steel at 3 plants. It takes 20 minutes to manufacture 1 ton of steel at plant 1, 16 minutes at plant 2, and 15 minutes at plant 3 (regardless of the type of steel made). Each plant is open 40 hours a week, and each week 100 tons of each type of steel must be produced. The production costs of a ton of steel at each of the 3 plants is given below.

<table>
<thead>
<tr>
<th></th>
<th>Steel 1</th>
<th>Steel 2</th>
<th>Steel 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant 1</td>
<td>$60</td>
<td>$40</td>
<td>$28</td>
</tr>
<tr>
<td>Plant 2</td>
<td>$50</td>
<td>$30</td>
<td>$30</td>
</tr>
<tr>
<td>Plant 3</td>
<td>$43</td>
<td>$20</td>
<td>$20</td>
</tr>
</tbody>
</table>

(a). Formulate a Balanced Transportation problem to minimize the cost of meeting the weekly demand requirements. Represent your formulation as a transportation tableau (cost and requirement table).

(b). As we know, the Balanced Transportation problem is a special case of a Transshipment problem. Find a feasible tree solution for the Transshipment problem of part (a).

(c). Suppose the time required to produce 1 ton of steel depends on the type of steel as well as on the plant (see table below). Could a Balanced Transportation problem still be formulated? If so, give the cost and requirement table. If not, explain why, and give a formulation as an LP.

<table>
<thead>
<tr>
<th>Time (in minutes)</th>
<th>Steel 1</th>
<th>Steel 2</th>
<th>Steel 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant 1</td>
<td>15</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>Plant 2</td>
<td>15</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>Plant 3</td>
<td>10</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

2). Consider two nonempty polyhedra $P = \{ x \mid Ax \leq b \}$ and $Q = \{ x \mid Dx \leq d \}$. We wish to decide if the intersection of these two polyhedra is nonempty (i.e., we want to know whether $P \cap Q \neq \emptyset$).

(a). State an LP such that if $P \cap Q \neq \emptyset$ the optimal solution to your LP is in the intersection, and if the intersection is empty, then the LP is infeasible.

(b). State the dual of the LP you gave in part (a).

(c). Suppose that the intersection is empty. Show that there exists a vector $c$ such that $cx < cx'$ for all $x \in P$ and all $x' \in Q$. Hint: use the dual.

3). Consider the Simplex method applied to an LP in standard form, $\min \{ cx \mid Ax = b, \ x \geq 0 \}$ where the rows of $A$ are linearly independent. For each of the following, either give a simple proof, or a small counterexample.

(a). Let $x'$ be a BFS obtained by a single pivot from BFS $x$, where $x \neq x'$, then $cx' < cx$.

(b). A variable that has just left the basis cannot reenter the basis in the very next iteration.

(c). A variable that has just entered the basis cannot leave the basis in the very next iteration.

4). Let $\{X_n, n \geq 0\}$ be a time-homogeneous discrete-time Markov chain with a countable state space $S$. Prove or give a counterexample to the following statement:

For an arbitrary function $f : S \rightarrow S$, the stochastic process $\{Y_n, n \geq 0\}$, where $Y_n = f(X_n)$, is also a Markov chain.

5). Consider a single-server finite-capacity exponential queue with total capacity $N$ (including the server), arrival intensity $\lambda$, and service intensity $\mu$. (This is an $M/M/1/N$ queue.) For each customer served, the system managers earn $r$ dollars. If a customer is rejected (because the system is at full capacity $N$), they
lose \( c \) dollars (due to the bad press for turning away a customer). If there are \( n \) customers in the system, they experience a (holding) cost of \( nq \) dollars per unit time. How much money per unit time are the system managers earning on average? (The actual number may be negative, if they are losing money.)

6). A queueing system consists of two identical and independent servers. Each server has its unlimited waiting space. Customers arrive according to a Poisson process with intensity \( \lambda \). For each server, service times are iid random variables with first and second moments \( E[S] \) and \( E[S^2] \), respectively. Each arrival independently goes to server 1 with probability \( p \) and to server 2 with probability \( 1 - p \). After the service is complete, the customer leaves the system. Find the average customer waiting time \( W \) for this system.

7). The FFT Company wishes to come up with a production plan for the next six weeks. Each job lasts several weeks and once started must be carried out without interruption. During each week, a certain number of skilled workers are required to work full-time on the job. The following table presents the data:

<table>
<thead>
<tr>
<th>Job</th>
<th>Length</th>
<th>Week 1</th>
<th>Week 2</th>
<th>Week 3</th>
<th>Week 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>−</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>9</td>
<td>2</td>
<td>3</td>
<td>−</td>
</tr>
</tbody>
</table>

For instance, Job 1 lasts 3 weeks and requires 2 workers in the first week (the week it starts), 3 workers in the next week, and 1 worker in the last week (the week of its completion).

(a) Formulate an integer programming problem to find a schedule that minimizes the maximum number of workers during any of the six weeks.

(b) Suppose that Job 1 and Job 2 cannot be carried out simultaneously since they need the same machine. Reformulate the problem.

8). Consider the generic nonlinear optimization problem

\[
\begin{align*}
(P) \quad \min_x & \quad f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,
\end{align*}
\]

where \( f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^m, \) and \( h : \mathbb{R}^n \to \mathbb{R}^p. \) Define a function \( \phi : \mathbb{R}^m \to \mathbb{R} \) as

\[
\phi(u) := \begin{cases} 
\min_x \left\{ f(x) + u^T g(x) \right\} & \text{subject to } h(x) = 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

Note that \( \phi(u) \) is defined as the optimal value of another optimization problem if \( u \geq 0. \) Let \( \alpha \) denote the optimal value of (P) (including possibly \( \pm \infty \)).

(a) Show that \( \phi(u) \) is a concave function of \( u \) (or equivalently, \( -\phi(u) \) is a convex function of \( u \).)

(b) Show that \( \max_u \phi(u) \leq \alpha. \)

(c) Suppose that \( u^* \geq 0. \) Let \( x^* \) be an optimal solution for the problem defining \( \phi(u^*) \) such that \( g(x^*) \leq 0 \) and \( u^{*T} g(x^*) = 0. \) Show that \( x^* \) then is a global minimum for (P).
9). Consider the probability density function (pdf) of a random variable, $X$, given by $f(x) = K + \varepsilon \sin(2\pi x)$ for $0 \leq x \leq 1$ (0 otherwise).

(a) What constraints are there on the values of $K$ and $\varepsilon$?

(b) Find the proportion of values over 0.5 (i.e., $P(X > 0.5)$), find the mode of the pdf, and find the values of $K$ and $\varepsilon$ that will maximize the mean, $E(X)$.

(c) Give two different algorithms for generating samples from $f$, one that uses exactly one random number per random variate and the other that uses more than one random number per variate.

(d) Provide another algorithm similar to the acceptance-rejection scheme that has 100% efficiency and uses exactly two random numbers to produce each sample (but different from composition). Prove that your algorithm is correct. (Hint: Perhaps it could be called the acceptance-relocation scheme.)

10). Consider a D/M/1 queue in which arrivals occur every 1 minute. Assuming the system starts empty (the first arrival is at time $t = 1$ minute) and the mean service time is 1 second, we want to estimate the probability that a customer has to wait (any amount of time) upon arrival. For simplicity, we only consider the second customer.

(a) Calculate the probability analytically. What would you expect if you estimate the probability based on 1000 replications by using crude Monte Carlo simulation?

(b) If we use importance sampling with mean service time of 1 minute, give an unbiased estimator for the target probability.

(c) Explain how you would carry out importance sampling (b) for estimating the probability that the $n$th ($n > 2$) customer has to wait. Give an unbiased estimator.

11). A set $S$ of $n$ line segments in the plane is said to have a Type 1 degeneracy if all segments lie on a common line. They are said to have a Type 2 degeneracy if some subset of 3 or more endpoints are collinear (thus, if two segments share an endpoint, then there is a Type 2 degeneracy, but there are other cases of Type 2 degeneracies as well).

(a). How efficiently (in big-Oh notation, as a function of $n$) can one determine if $S$ has a Type 1 degeneracy? Explain briefly.

(b). How efficiently can one determine if $S$ has a Type 2 degeneracy? Explain briefly.

(c). Now assume that $S$ has no Type 1 or Type 2 degeneracies. Consider the arrangement, $A(S)$, of the $n$ line segments. Let $v$, $e$, and $f$ denote the number of vertices, edges, and faces of $A(S)$. Give upper bounds (as an explicit function of $n$, not big-Oh) for $v$, $e$, and $f$. (Try to make your upper bounds as good as possible.)

12). (a). Given a set $S_1$ of $n$ blue points in $\mathbb{R}^2$ and a set $S_2$ of $m$ red points in $\mathbb{R}^2$, let $\Sigma$ be the set of $mn$ line segments connecting each point of $S_1$ with each point of $S_2$.

Sketch an efficient method of determining if there is a stabbing line $\ell$ that stabs (crosses) every segment of $\Sigma$. What is the running time of your method?

(b). Now let $\Sigma$ be an arbitrary set of $k$ line segments in the plane. Describe (briefly) a data structure that allows one to answer the following query efficiently: For a query line $\ell$, determine how many segments of $\Sigma$ are crossed by $\ell$.

(i). Preprocessing time is $O(\quad)$
(ii). Storage space (memory usage) is $O(\quad)$

(iii). Query time is $O(\quad)$

13). (a). We are given an undirected, connected graph $G = (V, E)$ with (nonnegative) costs on the edges $c_{ij}$. A balanced spanning tree is defined as a spanning tree of $G$ if, from among all spanning trees of $G$, the difference between the maximum edge cost and the minimum edge cost is as small as possible. Describe a polynomial-time algorithm for finding a balanced spanning tree. Make sure to state the running time of your algorithm.

(b). We are given a complete graph $G$ with (nonnegative) costs on the edges $c_{ij}$. For every Hamilton cycle of $G$, we define its balance as the difference between the maximum edge cost and the minimum edge cost in the tour. The minimum balance tour problem asks for a tour of minimum balance among all Hamilton cycles. Show that this problem is NP-hard by giving a reduction from one of the NP-complete problems discussed in class. (You do not need to show that the problem is in NP.)

14). Let $G = (N, E)$ be an undirected (connected) graph with $n$ nodes and $m$ edges. Define a matching $M$ to be maximal if for every edge $e \in E \setminus M$, $M \cup e$ is not a matching.

(a). Show how to find a maximal matching in time $O(m)$.

(b). Let $M^*$ be a maximum cardinality matching. Show that $|M| \geq 0.5|M^*|$. 

15). Consider a Markov Decision Process with the state space $X = \{0, 1\}$, the action sets $A(0) = A(1) = A = \{b, c\}$, transition probabilities $p(x|a, a) = 1$ for all $x \in X$ and all $a \in A$. The discount factor $\alpha$ equals 0.5. Let $\mu$ be the initial state distribution with $\mu(0) = \mu(1) = 1/2$ and $\pi$ be the randomized stationary policy with $\pi(b|0) = \pi(c|0) = \pi(b|1) = \pi(c|1) = 0.5$.

Recall that for a policy $\sigma$, the occupation measure is a vector $(Q^\sigma_\mu(0, b), Q^\sigma_\mu(0, c), Q^\sigma_\mu(1, b), Q^\sigma_\mu(1, c))$ with

$$Q^\sigma_\mu(x, a) = \sum_{t=0}^{\infty} \alpha^t P^\sigma_\mu \{x_t = x, a_t = a\}, \quad x \in X, \ a \in A.$$ 

Find a vector $(\gamma_1, \gamma_2, \gamma_3)$ with nonnegative elements such that $\gamma_1 + \gamma_2 + \gamma_3 = 1$ and find three (nonrandomized) stationary policies $\phi^1, \phi^2, \phi^3$ such that

$$Q^\sigma_\mu = \sum_{i=1}^{3} \gamma_i Q^\sigma_{\phi_i}. \quad (1)$$

Remark. This problem has multiple solutions. Find at least one of them.

16). Consider a Markov Decision Process with three states $\{1, 2, 3\}$ and action sets $\{a, b\}$ in each of these states. The transition probabilities $p(y|x, a)$ are defined by the matrix

$$(P^a(x, y)) = \begin{pmatrix}
0.5 & 0.5 & 0 \\
0.25 & 0.25 & 0.5 \\
0.5 & 0 & 0.5
\end{pmatrix}$$

and similarly $p(y|x, b)$ are defined by the matrix

$$(P^b(x, y)) = \begin{pmatrix}
0.5 & 0.25 & 0.25 \\
0 & 0 & 1 \\
0.6 & 0.3 & 0.1
\end{pmatrix}$$

The one-step rewards are $(r(1, a), r(2, a), r(3, a)) = (1, 4, 2)$ and $(r(1, b), r(2, b), r(3, b)) = (3, 2, 1)$.

Consider average rewards per unit time. Write a linear program (LP) that will allow you to construct a stationary optimal policy. Explain how you will derive the optimal policy after you solve the LP (you are not required to solve the LP).