Qualifying Exam (February 2011): Quantitative Finance

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.
Do 2 out of problems 1,2,3.
Do 2 out of problems 4,5,6.
Do 3 out of problems 7,8,9,10,11,12.

All problems are weighted equally. **On this cover page write which seven problems you want graded.**

problems to be graded:

______________________________________________

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Name (PRINT CLEARLY), ID number**

______________________________________________

**Signature**

______________________________________________
1). Consider one share of a convertible bond with underlying stock price $S_t$ at time $t$ and maturity $T$ which gives the holder right to receive at expiration either one share of the underlying stock or a fixed cash payment of $K$, whatever is larger. Assume that (1) the bond has no coupon, (2) the holder will sell the stock immediately if she exchanges the bond to the underlying stock at time $T$, (3) the default probability of the bond is zero, and (4) the risk free rate $r$ is constant with continuous compounding. Let $V(t)$ denote the price of the bond. Adapt the Black-Scholes option pricing formula for European options to price the convertible bond.

2). The stock price process $(S_t)_{t \geq 0}$ is modeled by the Itô stochastic differential equation as

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where $(B_t)_{t \geq 0}$ is the standard Brownian Motion and $\mu$ and $\sigma$ are constant positive real numbers. Suppose that the underlying stock is a dividend paying stock and $(D_t)_{t \geq 0}$ is the process of cumulative dividends. The price process of the stock and the cumulative dividend is defined by the Itô process

$$d(S_t + D_t) = (\mu + d)S_t dt + \sigma S_t dB_t$$

where $d$ is the continuous compounding dividend rate. Let $r$ be the risk-free rate. Deduce step by step the Black-Scholes partial differential equation under the complete market.

3). We would like to price a put option with the maturity $T$ and the strike price $K$ under the Gamma-subordinated geometric Brownian motion model (i.e. the Variance Gamma model). Let $(B_t)_{t \geq 0}$ be the standard Brownian Motion and $(\tau_t)_{t \geq 0}$ be the standard gamma process with $\tau_t \sim Gamma(\lambda t, \lambda)$. Suppose $(\tau_t)_{t \geq 0}$ and $(B_t)_{t \geq 0}$ are independent. The risk-neutral stock price process $(S_t)_{t \geq 0}$ is modeled by the Gamma-subordinated geometric Brownian motion as follows:

$$S_t = S_0 \exp \left( rt - \frac{\sigma^2}{2} \tau_t + \sigma B_{\tau_t} \right)$$

where $r$ is the risk-free rate and $\sigma$ are constant positive real numbers. Describe Monte Carlo analysis to determine the price of the option at time 0 using $N$ trials.

4). Let

$$P = \begin{pmatrix}
0 & 0.25 & a & b \\
0.25 & 0 & c & d \\
0 & 0.25 & 0 & e \\
0.75 & 0 & 0 & 0
\end{pmatrix}$$

be a transition matrix of an irreducible Markov chain with the limiting probability distribution $(0.25, 0.25, 0.25, 0.25)$. Find $a$, $b$, $c$, $d$, $e$, $f$, and $g$.

5). Two types of customers arrive to a single-server queue according to two independent Poisson processes $N_1(t)$ and $N_2(t)$ with intensities $\lambda_1$ and $\lambda_2$ respectively. Type $i$ customers arrive according to the Poisson Process $N_i(t)$, $i = 1, 2$. Service times for type $i$ customers are identical, deterministic, and equal to $T_i$, $i = 1, 2$. Waiting space is unlimited. Find the average waiting time (the average times between arrivals and departures) for type 1 customers.

6). For an M/G/1/loss system with arrival rate $\lambda$ and mean service time $1/\mu$, find the fraction of lost customers.

7). Consider a gambling in which we are tossing a coin, and when we get a “Head”, we win and get 1$ per one bet, but when we get a “Tail”, we loss and have to pay 1$ per one bet to our counter party. One gambler have a strategy that
• he starts with one bet of the gambling.
• the gambler double his bet after every loss.
• if he wins then he stops the gambling.

Hence, the first win would recover all previous losses plus win a profit equal to 1$. (1) Compute expected value of the strategy if the maximum number of tossing is \( N \). (2) Suppose the probability of “Tail” is greater than 1/2. Then the expected value of profit goes to negative infinite as \( N \) goes to infinite.

8) We would like to measure the value at risk (VaR) of our portfolio using the multi-factor model. Consider a portfolio with \( N \)-stocks and \( K \)-risk-factors. Let \( R_1, R_2, \cdots, R_N \) be random variables for returns of stocks in the portfolio and \( F_1, F_2, \cdots, F_K \) be random variables for factors. The capital allocation rate for the \( n \)th stock is denoted by \( w_n \) and the return of the portfolio is equal to \( R_p = \sum_{n=1}^{N} w_n R_n \). According to the multi-factor model the stock returns in the portfolio are given by

\[
R_n = \mu_n + \sum_{k=1}^{K} \beta_{n,k} F_k + \varepsilon_n, \quad n = 1, 2, \cdots, N,
\]

where \( \mu_n \) is a constant, \( \beta_{n,k} \) is the beta coefficient for the \( k \)th factor with respect to the \( n \)th stock, and \( \varepsilon_n, \quad n = 1, 2, \cdots, N \) are uncorrelated random variables. Calculate the VaR of the portfolio at the 1% confidence level under assumption that

\[
(F_1, F_2, \cdots, F_K) \sim N(0, \Sigma)
\]

and

\[
\varepsilon_n \sim N(0, \sigma_n^2), \quad n = 1, 2, \cdots, N.
\]

Note that \( F(-2.33) \approx 0.01 \) where \( F \) is the cumulative distribution function of the standard normal distribution.

9) The Average Value-At-Risk is a popular coherent risk measure defined by

\[
\text{AVaR}_\alpha(X) = \frac{1}{\alpha} \int_{0}^{\alpha} \text{VaR}_x(X) dx
\]

where \( X \) is the portfolio return. If the distribution of \( R \) is continuous then we have the following property

\[
\text{AVaR}_\alpha(X) = -E[X | X < -\text{VaR}_\alpha(X)] = \frac{1}{\alpha} E \left[ -X \mathbb{1}_{\{X < -\text{VaR}_\alpha(X)\}} \right]
\]

where

\[
\mathbb{1}_{\{X < -\text{VaR}_\alpha(X)\}} = \begin{cases} 1, & \text{if } X < -\text{VaR}_\alpha(X) \\ 0, & \text{if } X \geq -\text{VaR}_\alpha(X) \end{cases}
\]

Find closed form solution of \( \text{AVaR}_\alpha(X) \) if \( X \sim N(0, \sigma^2) \).

10) Consider a portfolio consists as \( N \) stocks. Optimize the portfolio using the mean-variance optimization by determining the Lagrangian equation determining the optimal portfolio weights. That is solve

\[
\min_w \text{var} \left( \sum_{n=1}^{N} w_n R_n \right)
\]

s.t.

\[
\sum_{i=1}^{N} w_i (\mu_i - r_f) = m - r_f
\]

\[
\sum_{i=1}^{N} w_i = 1
\]
where $\mu_i = E[R_i]$, $m$ is the expected portfolio return, $r_f$ is the benchmark return, and $R_i$ is the return of $i$-th asset.

11).

12).